PHILLIPS-LOOMIS MATHEMATICAL SERIES

ELEMENTS OF GEOMETRY

BY

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PREFACE

THE mathematical series of which this book is the first to be published is founded on the works of the late Professor Elias Loomis. In the present instance, however, the work can scarcely be called a revision. We have utilized many of the terse and accurate statements and definitions of the Loomis Geometry, and have aimed to maintain the high standard of that work for rigorous demonstrations, but, aside from these similarities, the arrangement and method of presentation are essentially new.

While the book speaks for itself, we would call attention to some of its most important features.

The *Introduction* presents in the shortest possible compass the general outlines of the science to be studied, and leads at once to the actual study itself.

The *definitions* are distributed through the book as they are needed, instead of being grouped in long lists many pages in advance of the propositions to which they apply. An alphabetical index is added for easy reference.

The *constructions* in the Plane Geometry are also distributed, so that the student is taught how to make a figure at the same time that he is required to use it in demonstration.

In the Geometry of Space, the figures consist of half-tone engravings from the *photographs of actual models* recently constructed for use in the class-rooms of Yale University. By the side of these models are skeleton diagrams for the student to copy.

Extensive use has been made of *natural* and *symmetrical* methods of demonstration. Such methods are used for deducing the formula for the sum of the angles of a triangle, for the sum of the exterior and interior angles of a polygon, for parallel lines, for the theorems on regular polygons, and for similar figures in both Plane and Solid Geometry.

The *theory of limits* is treated with rigor, and not passed over as self-evident.

Attention is also called to the theorems of *proportion*, the use of *corollaries* as *exercises* to supply the need of "inventional geometry," and the Introduction to Modern Geometry.

We would here express our grateful acknowledgments to all who have aided in the preparation of this book; to Miss Elizabeth H. Richards, whose successful experience in fitting students for college in Plane Geometry has rendered her criticisms and suggestions most valuable, to Mr. E. H. Lockwood, of the Sheffield Scientific School, whose skill as a draughtsman has been of essential service in the preparation of the figures, and to our colleagues, Messrs. W. M. Strong and Joseph Bowden, Jr. Mr. Strong has selected, for the most part, the exercises at the end of the book, and has written a large part of the Introduction to Modern Geometry. Mr. Bowden, whose large experience in teaching successive Freshman classes has given him an unusual equipment, has written a considerable portion of the Solid Geometry, and has examined critically the references and proof-sheets of the book.

ANDREW W. PHILLIPS,
IRVING FISHER.

YALE UNIVERSITY, *June*, 1896.

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SPECIAL TERMS

An **axiom** is a truth assumed as self-evident.

A **theorem** is a truth which becomes evident by a train of reasoning called a **demonstration**.

A theorem consists of two parts, the *hypothesis*, that which is given, and the *conclusion*, that which is to be proved.

A **problem** is a question proposed which requires a solution.

A **proposition** is a general term for either a theorem or problem.

One theorem is the **converse** of another when the conclusion of the first is made the hypothesis of the second, and the hypothesis of the first is made the conclusion of the second.

The converse of a truth is not always true. Thus, "If a man is in New York City he is in New York State," is true; but the converse, "If a man is in New York State he is in New York City," is not necessarily true.

When one theorem is easily deduced from another the first is sometimes called a **corollary** of the second.

A theorem used merely to prepare the way for another theorem is sometimes called a **lemma.**

SYMBOLS AND ABBREVIATIONS

$+$ plus.	Cons.—Construction.
$-$ minus.	Cor.—Corollary.
$>$ is greater than.	Def.—Definition.
$<$ is less than.	Fig.—Figure.
$\times$ multiplied by.	Hyp.—Hypothesis.
$=$ equals.	Iden.—Identical.
$\Leftrightarrow$ is equivalent to.	Q. E. D.—Quod erat demonstrandum.
Alt.-int.—Alternate interior.	Q. E. F.—Quod erat faciendum.
Ax.—Axiom.	Sup.-adj.—Supplementary adjacent.

GEOMETRY

INTRODUCTION

FUNDAMENTAL CONCEPTIONS

1. *Def.*—**Geometry** is the science of **space.**

2. Every one has a notion of space extending indefinitely in all directions. Every material body, as a rock, a tree, or a house, occupies a limited portion of space. The portion of space which a body occupies, considered separately from the matter of which it is composed, is a *geometrical solid.* The material body is a *physical solid.* Only geometrical solids are here considered, and they are called simply *solids.*

Def.—A **solid** is, then, a limited portion of **space.**

3. *Def.*—The boundaries of a solid are **surfaces** (that is, the surfaces separate it from the surrounding space).

A surface is no part of a solid.

4. *Def.*—The boundaries of a surface are **lines.**

A line is no part of a surface.

5. *Def.*—The boundaries (or ends) of a line are **points.**

A point is no part of a line.

6. The solid, surface, line, and point are the four fundamental conceptions of geometry. They may also be considered in the reverse order, thus:

(1.) A **point** has position but no magnitude.

(2.) If a point moves, it generates (traces) a **line.**
This motion gives to the line its only magnitude, *length.*

(3.) If a line moves (not along itself), it generates a **surface.**
This motion gives to the surface, besides length, *breadth.*

(4.) If a surface moves (not along itself), it generates a **solid.**
This motion gives to the solid, besides length and breadth, *thickness.*

Def.—A **figure** is any combination of points, lines, surfaces, or solids.

7. *Def.*—A **straight line** is a line which is the shortest path between any two of its points.

8. *Def.*—A **plane surface** (or simply a **plane**) is a surface such that, if any two points in it are taken, the straight line passing through them lies wholly in the surface.

9. *Def.*—Two straight lines are **parallel** which lie in the same plane and never meet, however far produced.

GEOMETRIC AXIOMS

10. All the truths of geometry rest upon three fundamental axioms, viz.:

(*a.*) **Straight line axiom.**—Through every two points in space there is one and only one straight line.

This is sometimes expressed as follows: Two points *determine* a straight line.

(*b.*) **Parallel axiom.**—Through a given point there is one and only one straight line parallel to a given straight line.

(*c.*) **Superposition axiom.**—Any figure in a plane may be freely moved about in that plane without change of size or shape. Likewise, any figure in space may be freely moved about in space without change of size or shape.

GENERAL AXIOMS

11. In reasoning from one geometric truth to another the following general axioms are also employed, viz.:

(1.) Things equal to the same thing are equal to each other.

(2.) If equals be added to equals, the wholes are equal.

(3.) If equals be taken from equals, the remainders are equal.

(4.) If equals be added to unequals, the wholes are unequal in the same order.

(5.) If equals be taken from unequals, the remainders are unequal in the same order.

(6.) If unequals be taken from equals, the remainders are unequal in the opposite order.

(7.) If equals be multiplied by equals, the products are equal; and if unequals be multiplied by equals, the products are unequal in the same order.

(8.) If equals be divided by equals, the quotients are equal; and if unequals be divided by equals, the quotients are unequal in the same order.

(9.) If unequals be added to unequals, the lesser to the lesser and the greater to the greater, the wholes will be unequal in the same order.

(10.) The whole is greater than any of its parts.

(11.) The whole is equal to the sum of all its parts.

(12.) If of two unequal quantities the lesser increases (continuously and indefinitely) while the greater decreases; they must become equal once and but once.

(13.) If of three quantities the first is greater than the second and the second greater than the third, then the first is greater than the third.

12. *Def.*—**Plane Geometry** treats of figures in the same plane.

13. *Def.*—**Solid Geometry,** or the **geometry of space,** treats of figures not wholly in the same plane.

PLANE GEOMETRY

BOOK I

FIGURES FORMED BY STRAIGHT LINES

14. *Defs.*—An **angle** is a figure formed by two straight lines diverging from the same point.

This point is the *vertex* of the angle, and the lines are its *sides*.

A clear notion of an angle may be obtained by observing the hands of a clock, which form a continually varying angle.

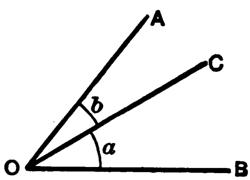


FIG. 1

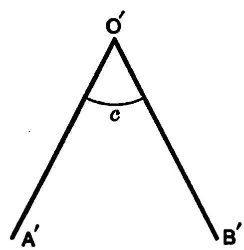


FIG. 2

We may designate an angle by a letter placed within as a and b in Fig. 1, and c in Fig. 2.

Three letters may be used, viz.: one letter on each of its sides, together with one at the vertex, which must be written between the other two, as AOC, BOC, and AOB in Fig. 1, and $A'O'B'$ in Fig. 2.

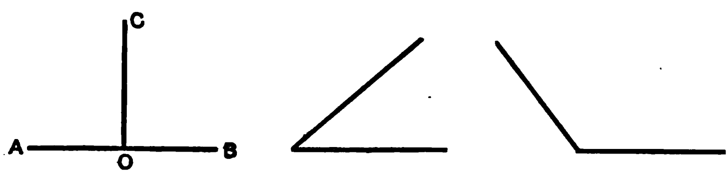
If there is but one angle at a point, it may be denoted by a single letter at that point, as O' in Fig. 2.

Angles with a common vertex and side, as a and b, are said to be **adjacent**.

15. *Def.*—Two angles are **equal** if they can be made to coincide. Also, in general, any two figures are equal which can be made to coincide.

Thus, suppose we place the angle AOB on the angle $A'O'B'$ so that O shall fall at O', and the side OA along $O'A'$; then, if the side OB also falls along $O'B'$, the angles are equal, *whatever may be the length of each of their sides.*

16. *Def.*—When one straight line is drawn from a point in another, so that the two adjacent angles are equal, each of these angles is a **right angle,** and the lines are **perpendicular.**



RIGHT ANGLES ACUTE ANGLE OBTUSE ANGLE

Thus, if the angles AOC and COB are equal, they are right angles, and CO is perpendicular to AB.

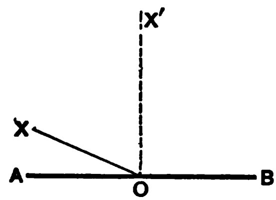
When a straight line is perpendicular to another straight line, its point of intersection with the second line is called the **foot of the perpendicular.**

17. *Def.*—An **acute** angle is an angle less than a right angle; an **obtuse** angle, greater.

The term **oblique** angle may be applied to any angle which is not a right angle.

PROPOSITION I. THEOREM

18. *From a point in a straight line one perpendicular, and only one, can be drawn (on the same side of the given straight line).*



GIVEN a straight line, AB, and any point, O, upon it.

TO PROVE—from O one, and only one, perpendicular can be drawn to AB (on the same side of AB).

Suppose a straight line OX to revolve about O. Ax. *c*

In every one of its successive positions it forms two different angles with the line AB, viz.: XOA and XOB.

As it revolves from the position OA around to the position OB the lesser angle, XOA, will continuously increase, and the other, XOB, will continuously decrease.

There must, therefore, be one and only one position of OX, as OX' where the angles become equal. Ax. 12

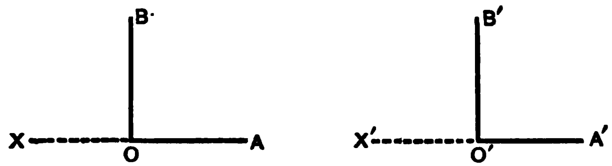
[If, of two unequal quantities, the lesser increases, etc.]

That is, there must be one and only one perpendicular to AB at O. Q. E. D.

Question.—The above proposition applies to the plane of the diagram. Could you draw any other lines perpendicular to AB at O *out* of the plane of the page?

PROPOSITION II. THEOREM

19. *All right angles are equal.*



GIVEN any two right angles AOB and $A'O'B'$.

TO PROVE they are equal.

Apply $A'O'B'$ to AOB so that the vertex O' shall fall on O, and so that A', any point in one side of $A'O'B'$, shall fall on some point in OA or OA produced.

Then the line $O'A'$ will coincide with OA, even if both be produced indefinitely. Ax. *a*

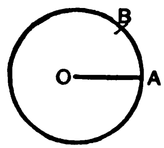
[Two points determine a straight line.]

If $O'B'$ should not fall along OB, there would be two lines, $O'B'$ and OB, perpendicular to the same line from the same point, which is impossible. § 18

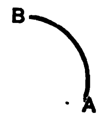
[From a point in a straight line, one perpendicular, and only one can be drawn.]

Therefore $O'B'$ must fall along OB—that is, the angles $A'O'B'$ and AOB coincide and are equal. Q. E. D.

20. *Defs.*—A **circle** is a figure bounded by a line all points of which are equally distant from a point within called the **centre.**



CIRCLE



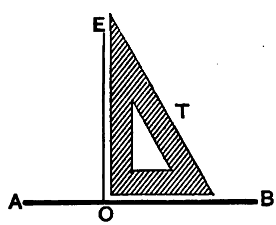
ARC

The bounding line is called the **circumference.**

Any portion of the circumference is called an **arc.**

Any one of the equal lines from the centre to the circumference (as OA) is called a **radius.**

21. CONSTRUCTION. *To draw a perpendicular from a straight line AB at some point in it, as O.*

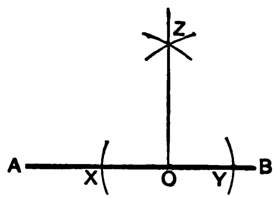


FIRST METHOD

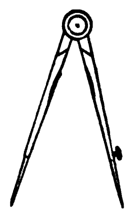
First method.—Place a right-angled ruler T with the vertex of its right angle at O and one of its edges along AB. Draw OE along its other edge. OE will be the required line for, first, it is drawn through O, and, second, it is drawn perpendicular to AB.

The student should observe that it is impossible to construct an absolutely accurate diagram, for no ruler is absolutely accurate nor can it be applied with absolute accuracy. Moreover the dots and marks formed by a pencil, however well sharpened, are not absolute points and lines, for the dots have *some* magnitude, and the marks *some* breadth. Diagrams only *approximate* the ideal points and lines intended.

If, however, the practical means employed *could be made perfect*, the resulting construction *would be* absolutely exact. Hence we may say of the preceding construction, the *method* is perfect, though the *means* can never be. This method is largely used by draughtsmen and carpenters.



SECOND METHOD



COMPASSES

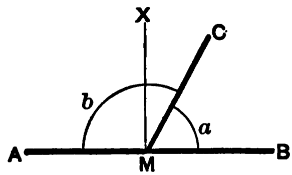
Second method (with straight ruler and compasses).—Take O as a centre, and with any convenient radius describe with the compasses two arcs cutting AB at X and Y. Then with X and Y as centres, with a somewhat longer radius describe two arcs cutting each other at Z. Join OZ with the ruler. OZ will be the perpendicular required.

[The correctness of the second method can be proved after reaching § 89.]

Of the two methods above described, the first has the advantage of quickness, but it assumes that the ruler is really made with a right angle, that is, it assumes that some one has already constructed a right angle and all we do is to copy it. The second method is free from this assumption, though, in both methods, it is assumed that the ruler is made with a straight edge, that is, that some one has already constructed a straight line. The first way of constructing a straight line was by stretching a string, a method still used by carpenters. In fact the word "straight" originally meant "stretched." The ancient Egyptians used this method, and even invented a way of making a right angle by stretching a cord. (See foot-note to § 317.)

PROPOSITION III. THEOREM

22. *The two angles which one straight line makes with another, upon one side of it, are together equal to two right angles.*



GIVEN—the straight line CM meeting the straight line AB at M and forming the angles a and b.

TO PROVE $a+b=2$ right angles.

Suppose MX drawn perpendicular to AB. § 18

[From a point in a straight line one perpendicular can be drawn.]

Then $BMX+XMA=2$ right angles. § 16

We may substitute for BMX its equal, $a+CMX$. Ax. 11

[A whole is equal to the sum of its parts.]

This gives $a+CMX+XMA=2$ right angles.

We may now substitute for $CMX+XMA$ the angle b.

[Same axiom.]

This gives $a+b=2$ right angles. Q. E. D.

23. *Defs.*—Two angles whose sum is equal to a right angle, are **complementary** angles.

Two angles whose sum is two right angles, are **supplementary** angles.

The two angles which one straight line makes with another on one side of it (as a and b), are **supplementary-adjacent** angles.

24. COR. I. *If one of the angles formed by the intersection of two straight lines is a right angle, the others are right angles.* (Fig. 1.)

Hint.—Apply Proposition III.

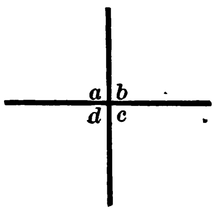


FIG. 1

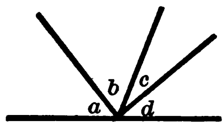


FIG. 2

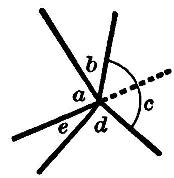


FIG. 3

25. COR. II. *If of two intersecting straight lines one is perpendicular to the other, then the second is also perpendicular to the first.*

Hint.—Apply Corollary I.

26. In COROLLARIES the proof is left, wholly or in part, to the student. Practice will give him the power of carefully stating and separating the steps and *finding for each a satisfactory reason.*

27. COR. III. *The sum of all the angles about a point on one side of a straight line equals two right angles.* (Fig. 2.)

Hint.—Group the angles into two angles and apply Proposition III.

28. COR. IV. *The sum of all the angles about a point equals four right angles.* (Fig. 3.)

Hint.—Prolong one of the lines through the vertex, separating the opposite angle *c* into two angles, and apply Corollary III.

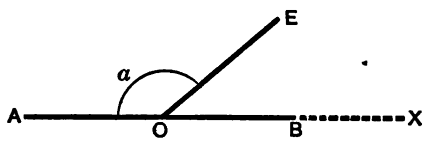
Question.—If, of three angles around a point, two are each one and a third right angles, how much is the third angle?

Question.—If six angles about a point are all equal, how large is each angle?

PROPOSITION IV. THEOREM

29. *If two adjacent angles are together equal to two right angles, their exterior sides are in the same straight line.*

[The converse of Proposition III.]



GIVEN $a+EOB=2$ right angles.

TO PROVE AO and OB form one straight line.

Let OX be the prolongation of AO.

$a+EOB=2$ right angles. Hyp.

$a+EOX=2$ right angles. § 22

[Being sup.-adj.]

Hence $a+EOB=a+EOX$. Ax. 1

Subtracting a, $EOB=EOX$. Ax. 3

Hence OB must coincide with OX.

Otherwise one of the angles (EOB and EOX) would include the other, and they could *not* be equal. Ax. 10

Therefore OB lies in the same straight line with OA.

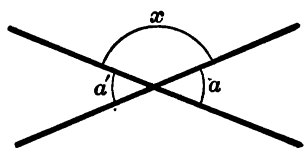
Q. E. D.

Question.—If two angles are supplementary-adjacent, and their difference is one right angle, how large is each?

Question.—The angles on the same side of a straight line are three in number. The greatest is three times the least, and the remaining one is twice the least. How large is each? In how many ways can they be arranged on the straight line?

PROPOSITION V. THEOREM

30. *If two straight lines intersect, the opposite (or vertical) angles are equal.*



GIVEN—two intersecting straight lines forming the opposite angles a and a'.

TO PROVE $a = a'$.

	$a + x = 2$ right angles.	§ 22
	$a' + x = 2$ right angles.	§ 22
	[Being, in each case, sup.-adj.]	
Therefore	$a + x = a' + x$.	Ax. 1
Subtracting x,	$a = a'$.	Ax. 3
		Q. E. D.

PARALLEL LINES AND SYMMETRICAL FIGURES

31. *Def.*—Two straight lines are **parallel** which lie in the same plane, but never meet, however far produced.

PARALLEL LINES

32. *Def.*—Two figures are **symmetrical with respect to a straight line** called an **axis of symmetry**, when, if one of them be folded over on that line as an axis, it will coincide with the other. (Fig. 1.)

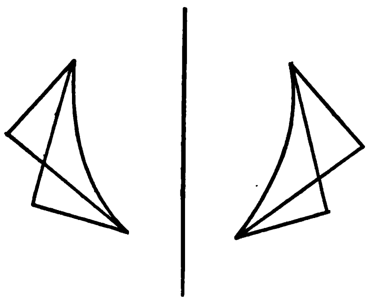


FIG. 1

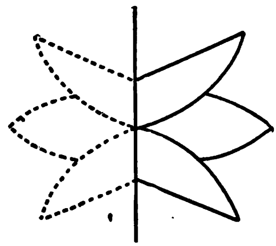
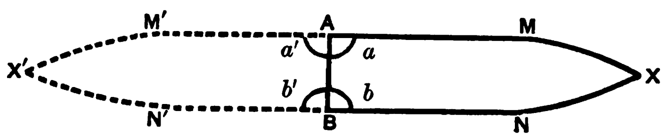


FIG. 2

A clear notion of this kind of symmetry may be obtained by drawing any figure in ink, and before the ink has dried folding the paper on to itself over a crease. The original figure and the resulting impression are symmetrical with respect to the crease as an axis. (Fig. 2.)

PROPOSITION VI. THEOREM

33. *Two straight lines perpendicular to the same straight line are parallel.*



GIVEN AM and BN perpendicular to AB.

TO PROVE AM and BN parallel.

If AM and BN should meet, either at the right or left, as at X, fold the figure AXB about AB as an axis to form the symmetrical impression $AX'B$, the right angles a and b forming the impressions a' and b' respectively.

Then AM and AM' form one and the same straight line,
and BN and BN' form one and the same straight line. § 29

[If two adjacent angles (as a' and a) are together equal to two right angles, their exterior sides are in the same straight line.]

Hence we would have two straight lines through X and X', which is absurd. Ax. *a*

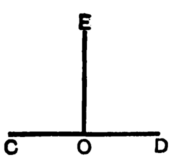
[Two points determine a straight line.]

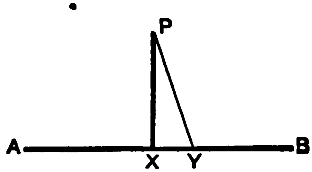
Therefore AM and BN cannot meet, and, as they lie in the same plane, they must be parallel. § 31

Q. E. D.

Question.—Will the preceding proposition still be true if the lines are not all confined to one plane?

34. COR. *Through a given point P without the line one and only one perpendicular can be drawn to a given straight line, AB.*

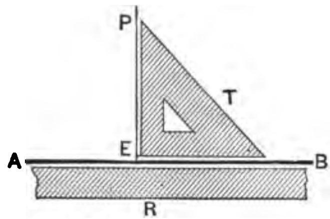




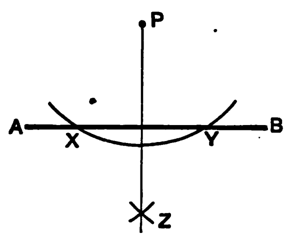
OUTLINE PROOF: From O in another line CD erect a perpendicular OE. (By what authority?) Superpose CD upon AB, and move it along AB until OE contains P. (What axiom applies?)

Second, suppose two were possible, as PX and PY, and show that this would contradict Proposition VI.

35. CONSTRUCTION. *To drop a perpendicular to a straight line AB from a point P without the line.*



First method.—Apply a straight edge of a ruler R to the straight line AB. Place one side of a right-angled ruler T upon the ruler R, making another side perpendicular to AB. Then slide T along AB until the perpendicular edge contains P. Draw PE along that edge. PE is the perpendicular required, for it is drawn through P and is perpendicular to AB.

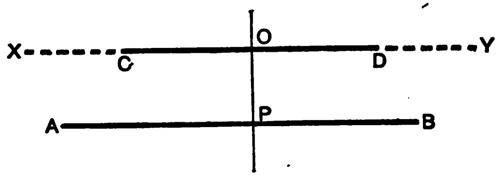


Second method.—From P as a centre with a convenient radius describe an arc cutting AB at X and Y. Then with X and Y in turn as centres describe arcs with equal radii intersecting at Z. Join PZ. This will be the required perpendicular.

[This can be proved correct after reaching § 104.]

PROPOSITION VII. THEOREM

36. *If two straight lines are parallel, and a third straight line is perpendicular to one of them, it is perpendicular to the other.* [Converse of Proposition VI.]



GIVEN—*CD* and *AB* parallel, and *PO* perpendicular to *AB*.

TO PROVE *PO* perpendicular to *CD*.

Suppose *XY* to be drawn through *O* perpendicular to *OP*.

Then *XY* is parallel to *AB*. § 33

[Two straight lines perpendicular to the same straight line are parallel.]

But *CD* is parallel to *AB*. Hyp.

Hence *CD* must coincide with *XY*. Ax. *b*

[Through any point there is one and only one straight line parallel to a given straight line.]

That is *CD* must be perpendicular to *PO*,

and *OP* is perpendicular to *CD*. § 25

Q. E. D.

37. CONSTRUCTION. *To draw a straight line through a given point C parallel to a given straight line AB.*

First method (Fig. 1).—Place a right-angled ruler in the position *T*, making one edge about the right angle coincident with *AB*, and along the other edge place a ruler *R*.

Then hold the ruler R firmly against the paper. Slide T to the position T' till its edge reaches C. Draw CX. It is the parallel required. (Why?)

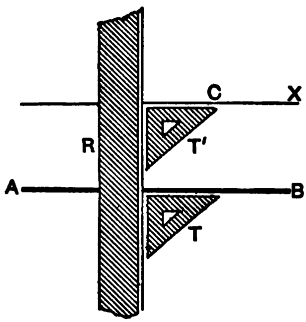


FIG. 1

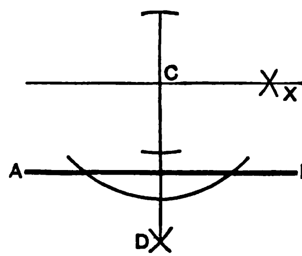


FIG. 2

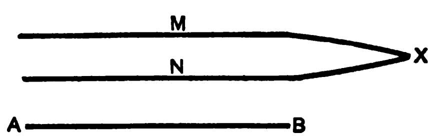
Second method (Fig. 2).—From C draw CD perpendicular to AB. § 35

At C draw CX perpendicular to CD. § 21

Then CX is the required parallel to AB. (Why?)

PROPOSITION VIII. THEOREM

38. ***If two straight lines are parallel to a third straight line, they are parallel to each other.***



GIVEN M and N each parallel to AB.

TO PROVE M and N parallel to each other.

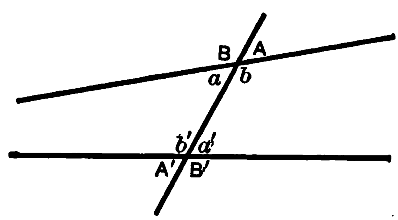
If M and N should meet, as at X, we would have two parallels to AB through the same point X, which is absurd. Ax. *b*

[Through one point there is one and only one straight line parallel to a given straight line.]

Therefore M and N cannot meet, and, lying in the same plane, must be parallel. § 31

Q. E. D.

39. *Defs.*—When two straight lines are cut by a third straight line, of the eight angles formed—



a, b, a', b', are **interior** angles.

A, B, A', B', are **exterior** angles.

a and a', or b and b', are **alternate-interior** angles.

A and A', or B and B', are **alternate-exterior** angles.

A and a', b and B', B and b', or a and A', are **corresponding** angles.

Question.—Of the eight angles, which are always equal, and why?

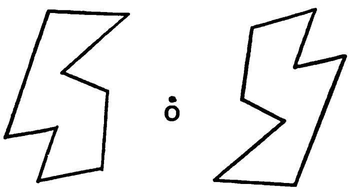
Question.—If $A = A'$, what other angles are also equal to A, and why? Are the remaining angles all equal, and if so, why?

Question.—If $A = A'$ and also $A = B$, what angles are equal, and why?

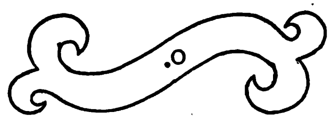
40. *Defs.*—Two figures are **symmetrical with respect to a point** called the **centre of symmetry** when, if one of them is revolved half way round on this point as a pivot, it will coincide with the other.

A single figure is said to be symmetrical with respect to a point called the centre of symmetry if, when the figure is turned half way round on this point as a pivot, each portion of the figure will take the position previously occupied by another part.

[A figure is said to be turned half way round a point when a line through the point turns through two right angles.]



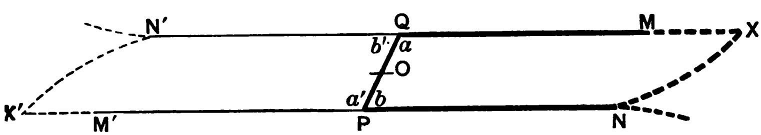
TWO FIGURES SYMMETRICAL WITH RESPECT TO O



A SINGLE FIGURE SYMMETRICAL WITH RESPECT TO O

PROPOSITION IX. THEOREM

41. *When two straight lines are cut by a third straight line, if the two interior angles on the same side of the cutting line are together equal to two right angles, then the two straight lines are parallel.*



GIVEN—*PQ* cutting *QM* and *PN* so that *a* and *b* on the same side of *PQ* are together equal to two right angles.

TO PROVE *QM* and *PN* parallel.

About O, the middle point of PQ, as a pivot, revolve the figure $QMXNP$ half way round to the symmetrical position $PM'X'N'Q$, so that P and Q exchange places.

The angle a is the supplement of b. Hyp.

Hence, when a takes the position a', PM' must be the prolongation of PN. § 29

[If two adjacent angles equal two right angles, their exterior sides form the same straight line.]

Likewise QN' is the prolongation of QM.

Now if these lines should meet on the right of PQ, as at X, they would also meet on the left, at X'. § 40

And we would have two straight lines between the two points, X and X', which is absurd. Ax. *a*

If they do *not* meet on the right of PQ, neither can they meet on the left of it. § 40

Hence QM and PN do not meet, and, being in the same plane, are parallel. Q. E. D.

It may be observed that the preceding proposition rests on only *two* of the three geometric axioms stated in § 10, viz.: the ***superposition axiom***, assumed in turning the figure unchanged about O, and the ***straight-line axiom***, used to prove that there cannot be two straight lines between X and X'. The *parallel axiom* (viz.: that through a point only one straight line can be drawn parallel to a given straight line) has only been used so far in Propositions VII. and VIII. Mathematicians have tried to dispense with the parallel axiom entirely, but have not succeeded. In fact, Lobatchewsky in 1829 proved that we can never get rid of the parallel axiom without assuming the space in which we live to be very different from what we know it to be through experience. Lobatchewsky tried to imagine a different sort of universe in which the parallel axiom would not be true. This imaginary kind of space is called ***non-Euclidean*** space, whereas the space in which we really live is called ***Euclidean***, because Euclid (about 300 B.C.) first wrote a systematic geometry of our space. In Lobatchewsky's space, Proposition IX. would be true, but Propositions VII. and VIII. would not be true, nor would §§ 47, 48, 49, 51, 58, etc., in Book I., and §§ 284, 327, 329, etc., in Book III.

42. CONSTRUCTION. *To bisect a given straight line, AB.*

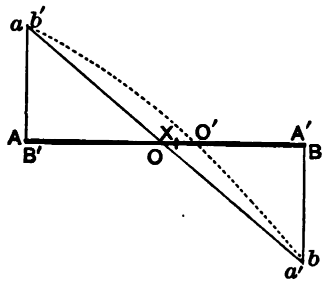


FIG. 1

First method (Fig. 1).—At A and B erect Aa and Bb equal perpendiculars on opposite sides of AB. Join ab cutting AB at O. O is the required middle point.

Proof.—Suppose the middle point of AB is not O, but some other point as X.

Then turn the whole figure about X until AX coincides with its equal BX, A falling on B (call this position of A, A'), and B on A (call this position of B, B'). And O will assume the position O' on the opposite side of X.

Then the perpendicular Aa will fall along Bb. § 18

[From a point in a straight line only one perpendicular can be drawn.]

And a will fall on b (call this position of a, a').

[Since Aa is equal to Bb.]

Likewise b will fall on a (call this position of b, b').

Then the straight line aOb takes the position $a'O'b'$.

That is, through two points, a and b, there would be two straight lines, which is absurd. Ax. *a*

Hence the supposition that O is not the middle point is false, and O is the middle point. Q. E. D.

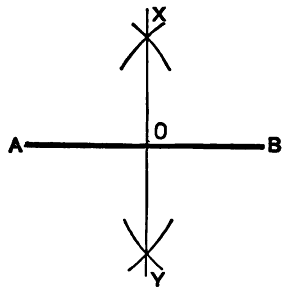


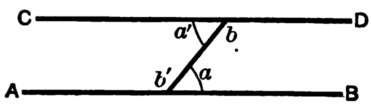
FIG. 2

Second method (Fig. 2).—From A and B as centres with the same radius describe arcs intersecting at X and Y. Join XY intersecting AB at O, the required middle point.

[This method can be proved correct after reaching § 104.]

PROPOSITION X. THEOREM

43. *If two straight lines are cut by a third straight line, making the alternate-interior angles equal, the lines are parallel.*



GIVEN $a = a'$.

TO PROVE AB and CD parallel.

$a' + b = 2$ right angles. § 22

[Being sup.-adj.]

Substitute for a' its equal a.

Then $a + b = 2$ right angles.

Therefore AB is parallel to CD. § 41

[When two straight lines are cut by a third straight line, if the two interior angles on the same side of the cutting line are together equal to two right angles, then the two straight lines are parallel.] Q. E. D.

44. COR. I. *If two or more straight lines are cut by a third, so that corresponding angles are equal, the straight lines are parallel.*

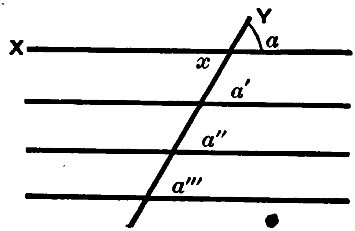


FIG. 1

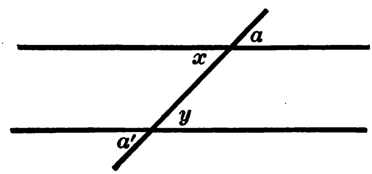
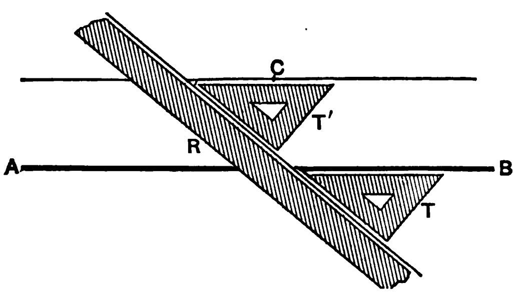


FIG. 2

Hint.—Reduce to Proposition X. by means of Proposition V.

45. COR. II. *If two straight lines are cut by a third straight line so that the alternate-exterior angles are equal, the lines are parallel.*

Hint.—Reduce to Proposition X. by Proposition V.

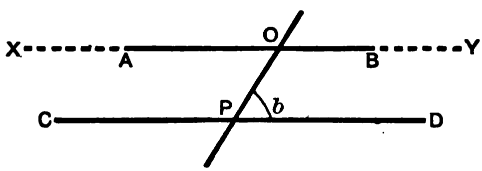


46. *Exercise.*—Show by § 44 that the construction of § 37 may be effected as in the preceding figure.

PROPOSITION XI. THEOREM

47. *If two parallel lines are cut by a third straight line, the sum of the two interior angles on the same side of the cutting line is two right angles.*

[Converse of Proposition IX.]



GIVEN—AB and CD parallel and cut by the straight line OP.

TO PROVE $b+POB=2$ right angles.

Suppose XY to be a line drawn through O, making

$b+POY=2$ right angles.

Then XY is parallel to CD. § 41

[When two straight lines are cut by a third straight line, if the two interior angles on the same side of the cutting line are together equal to two right angles, the two straight lines are parallel.]

But AB is parallel to CD. Hyp.

Hence AB coincides with XY. Ax. b

[Through a given point only one straight line can be drawn parallel to a given straight line.]

And $POB=POY$. Coinciding

Hence $b+POB=b+POY$. Ax. 2

But $b+POY=2$ right angles. Cons.

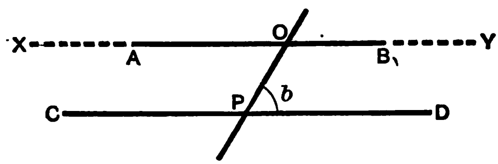
Hence $b+POB=2$ right angles. Ax. 1

Q. E. D.

PROPOSITION XII. THEOREM

48. *If two parallel lines are cut by a third straight line, then the alternate-interior angles are equal.*

[Converse of Proposition X.]



GIVEN AB and CD parallel.

TO PROVE $b=AOP$.

Suppose XY to be a line drawn through O, making $XOP=b$.

Then XY is parallel to CD. § 43

[If two straight lines are cut by a third straight line, making the alternate-interior angles equal, the lines are parallel.]

But AB is parallel to CD. Hyp.

Hence AB coincides with XY. Ax. *b*

And $AOP=XOP$. Coinciding

But $b=XOP$. Hyp.

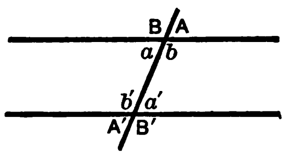
Therefore $AOP=b$. Ax. 1

Q. E. D.

49. COR. *If two or more parallel lines are cut by a third straight line, the corresponding angles are equal.*

Hint.—Reduce to Proposition XII.

50. *Remark.*—It follows from the previous propositions and corollaries that if two lines are parallel and cut by a third straight line, as in the figure,

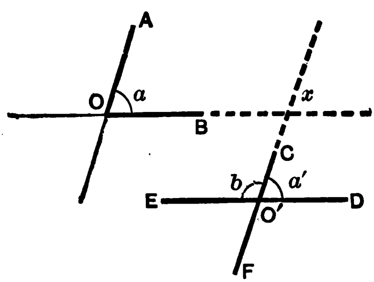


then $$A=a=a'=A',$$
$$B=b=b'=B',$$

and any angle of the first set is supplementary to any angle of the second set.

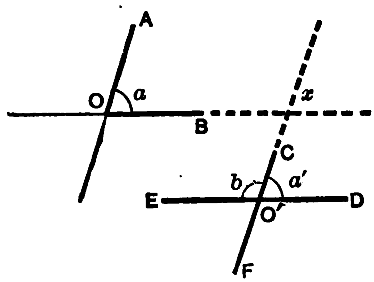
PROPOSITION XIII. THEOREM

51. *Two angles whose sides are parallel, each to each, are either equal or supplementary.*



GIVEN—the angles at O and O' with their sides OA and OB respectively parallel to CF and ED.

TO PROVE the angle $a=a'$, and $a+b=2$ right angles.



Produce OB and $O'C$ until they intersect.

Then $\left.\begin{array}{l} a=x \\ a'=x \end{array}\right\}$ § 49

[Being corresponding angles of parallel lines.]

Therefore $a=a'$. Ax. I

Moreover, $a'+b=2$ right angles. § 22

Substituting a for its equal a',

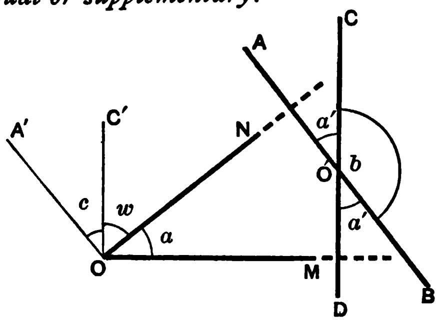
$a+b=2$ right angles. Q. E. D.

52. *Remark.*—To determine when the angles are equal and when supplementary, we observe that every angle, viewed from its vertex, has a *right* and a *left* side. (Thus OA is the left side of a.) Now, if the two angles have the right side of one parallel to the right side of the other and likewise their left sides parallel, they are equal; whereas, if the right side of each is parallel to the left side of the other, they are supplementary. Or, briefly, if their parallel sides are in the *same* right-and-left *order*, they are equal, if in *opposite order*, supplementary.

Thus, a and $EO'F$, which have their sides parallel, right to right (OB to $O'E$) and left to left (OA to $O'F$), are equal, while a and $EO'C$, which have their sides parallel right to left (OB to $O'E$) and left to right (OA to $O'C$), are supplementary. The student can easily test and verify all the sixteen cases obtained by comparing each of the four angles about O with each of the four about O'.

PROPOSITION XIV. THEOREM

53. ***Two angles whose sides are perpendicular, each to each, are either equal or supplementary.***



GIVEN—the angle *NOM*, or *a*, and the lines *AB* and *CD* intersecting at *O′* and respectively perpendicular to *ON* and *OM*.

TO PROVE—the angle $a=a'$, and $a+b=2$ right angles.

At *O*, draw *OA′* parallel to *AB* and *OC′* parallel to *CD*.

OA′, being parallel to *AB*, is perpendicular to *ON*. § 36

[If two straight lines are parallel, and a third straight line is perpendicular to one of them, it is perpendicular to the other.]

For the same reason *OC′*, being parallel to *CD*, is perpendicular to *OM*.

From each of the right angles *A′ON* and *C′OM* take away the common angle *w*.

This leaves $c=a$. Ax. 3

But $c=a'$. § 51

[Having their sides respectively parallel, and in the same right-and-left order.]

Therefore $a=a'$. Ax. 1

Moreover $a'+b=2$ right angles. § 22

[Being supplementary-adjacent.]

Substituting *a* for its equal *a′*,

$a+b=2$ right angles. Q. E. D.

54. *Remark.*—The angles are equal if their sides are perpendicular right to right and left to left, but supplementary if their sides are perpendicular in opposite right-and-left order.

Thus a and $DO'B$, which have their right sides (OM and $O'D$) perpendicular and their left sides (ON and $O'B$) perpendicular, are equal; etc., etc.

TRIANGLES

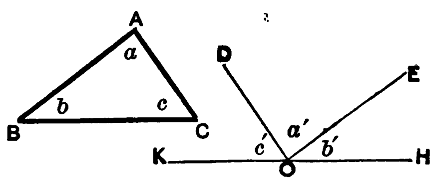
55. *Def.*—A **triangle** is a figure bounded by three straight lines called its **sides.**

56. *Def.*—A **right** triangle is a triangle one of whose angles is a right angle.

57. *Def.*—An **equiangular** triangle is one whose angles are all equal.

PROPOSITION XV. THEOREM

58. *The sum of the three angles of any triangle is two right angles.**



GIVEN ABC, any triangle, with a, b, and c its angles.

TO PROVE $a+b+c=2$ right angles.

Draw KH parallel to BC, and from O, any point of this line, draw OE and OD parallel respectively to the sides AB and AC.

* This was first proved by Pythagoras or his followers about 550 B.C.

Then
$$\left.\begin{array}{l} a=a' \\ b=b' \\ c=c' \end{array}\right\} \qquad \S\ 51$$

[Having their sides parallel and in the same right-and-left order.]

Hence $\quad a+b+c=a'+b'+c'$. $\quad$ Ax. 2

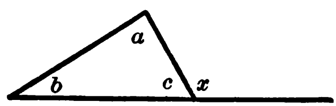
But $\quad a'+b'+c'=2$ right angles. $\quad$ § 27

[The sum of all the angles about a point on one side of a straight line equals two right angles.]

Hence $\quad a+b+c=2$ right angles. $\quad$ Ax. 1

Q. E. D.

59. COR. I. *If one side of a triangle be produced, the exterior angle thus formed equals the sum of the two opposite interior angles (and hence is greater than either of them).*



OUTLINE PROOF: $a+b+c=2$ right angles $=x+c$, whence $a+b=x$.
[Give reasons.]

60. COR. II. *If the sum of two angles of a triangle be given, the third angle may be found by taking the sum from two right angles.* [What axiom applies?]

61. COR. III. *If two angles of one triangle are equal respectively to two angles of another triangle, the third angles will be equal.* [What two axioms apply?]

62. COR. IV. *A triangle can have but one right angle, or one obtuse angle.*

63. COR. V. *In a right triangle the sum of the two angles besides the right angle is equal to one right angle.*

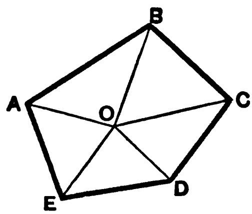
64. COR. VI. *In an equiangular triangle, each angle is one-third of two right angles, and hence two-thirds of one right angle.*

65. *Defs.*—A **polygon** is a figure bounded by straight lines called its **sides.**

A polygon is **convex,** if no straight line can meet its sides in more than two points.

PROPOSITION XVI. THEOREM

66. *The sum of all the angles of any polygon is twice as many right angles as the figure has sides, less four right angles.*



GIVEN $ABCDE$, any polygon, having n sides.

TO PROVE—the sum of its angles is $2n-4$ right angles.

From any point O within the polygon draw lines to all the vertices forming n triangles.

The sum of the angles of each triangle is equal to 2 right angles. § 58

Hence the sum of the angles of the n triangles is equal to $2n$ right angles.

But the angles of the polygon make up all the angles of all the triangles except the angles about O, which make 4 right angles. § 28

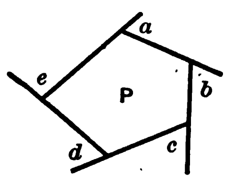
Hence the sum of the angles of the polygon is $2n-4$ right angles. Q. E. D.

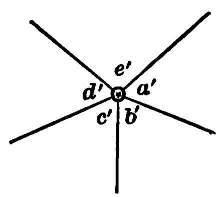
67. *Defs.*—A **quadrilateral** is a polygon of four sides, a **pentagon,** of five, a **hexagon,** of six, an **octagon,** of eight, a **decagon,** of ten, a **dodecagon,** of twelve, a **pentedecagon,** of fifteen.

68. *Exercise.*—The sum of the angles of a quadrilateral equals what? of a pentagon? of a hexagon?

PROPOSITION XVII. THEOREM

69. *If the sides of any polygon be successively produced, forming one exterior angle at each vertex, the sum of these exterior angles is four right angles.*





GIVEN—the polygon P with successive exterior angles a, b, c, d, e.

TO PROVE $a+b+c+d+e=4$ right angles.

Through any point O draw lines successively parallel to the sides produced.

Then
$$\left.\begin{array}{c} a=a' \\ b=b' \\ c=c' \\ \text{etc.} \end{array}\right\} \qquad \S\ 51$$

[Two angles are equal if their sides are parallel and in the same order.]

Hence $a+b+c+\text{etc.}=a'+b'+c'+\text{etc.}$ Ax. 2

But $a'+b'+c'+\text{etc.}=4$ right angles. § 28

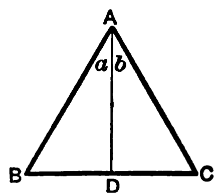
Therefore $a+b+c+\text{etc.}=4$ right angles. Ax. 1

Q. E. D.

70. *Defs.*—An **isosceles** triangle is a triangle two of whose sides are equal. The third side is called the **base.** The opposite vertex is called the **vertex** of the isosceles triangle, and the angle at that vertex the **vertex angle.** An **equilateral** triangle is one whose **three** sides are equal.

PROPOSITION XVIII. THEOREM

71. *The angles at the base of an isosceles triangle are equal.*



GIVEN—the isosceles triangle ABC, AB and AC being equal sides.

TO PROVE the angle B equals the angle C.

Suppose AD to be a line bisecting the angle A.

On AD as an axis revolve the figure ADC till it falls upon the plane of ADB.

AC will fall along AB.

[Since angle $a = b$, by construction.]

C will fall on B.

[Since $AB = AC$, by hypothesis.]

DB will coincide with DC. Ax. *a*

[Their extremities being the same points.]

Hence angle $B =$ angle C. § 15

[Since they coincide.] Q. E. D.

72. COR. I. *The line which bisects the vertex angle of an isosceles triangle bisects the base.*

Hint.—Show where this was proved in the preceding demonstration.

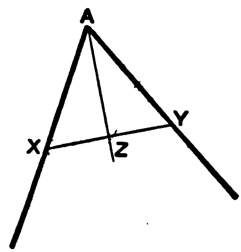
73. COR. II. *The line joining the middle point of the base with the vertex of an isosceles triangle bisects the vertex angle.*

Hint.—If not, draw the line which *does* bisect the vertex angle and prove it coincides with the given line.

74. COR. III. *Every equilateral triangle is also equiangular, and each angle is one-third of two right angles.*

Question.—In how many different ways is an equilateral triangle isosceles?

75. CONSTRUCTION. *To bisect any given angle A.*

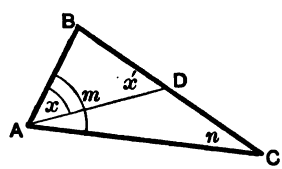


On the sides of the angle, lay off $AX = AY$. Join XY. Bisect XY at Z (§ 42). Join AZ. AZ will bisect the angle A. The student may prove this method correct.

Hint.—Apply one of the preceding corollaries.

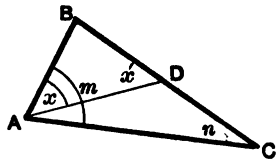
PROPOSITION XIX. THEOREM

76. *If two sides of a triangle are unequal, the opposite angles are unequal, and the greater angle is opposite the greater side.*



GIVEN in the triangle ABC the side $BC >$ side AB.

TO PROVE the angle $m >$ angle n.



On BC take $BD=BA$, and join AD.

Then $x=x'$. § 71

[Being base angles of an isosceles triangle.]

But $x'>n$. § 59

[An exterior angle of a triangle (ADC) is greater than either of the opposite interior angles.]

Substituting x for x', $x>n$.

But $m>x$. Ax. 10

Hence $m>n$. Ax. 13

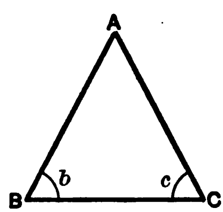
Q. E. D.

OUTLINE PROOF: $m>x=x'>n$, hence $m>n$.

PROPOSITION XX. THEOREM

77. *If two angles of a triangle are equal, the sides opposite are equal—that is, the triangle is isosceles.*

[Converse of Proposition XVIII.]



GIVEN in the triangle ABC, the angle $b=c$.

TO PROVE side $AC=$side AB.

If AC and AB were unequal, b and c would be unequal. § 76

[If two sides of a triangle are unequal the opposite angles are unequal, etc.]

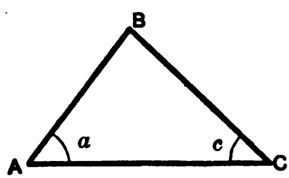
But this contradicts the hypothesis that angle b = angle c.

Hence $AC = AB$. Q. E. D.

PROPOSITION XXI. THEOREM

78. *If two angles of a triangle are unequal, the opposite sides are unequal, and the greater side is opposite the greater angle.*

[Converse of Proposition XIX.]



GIVEN in the triangle ABC, the angle $a >$ angle c.

TO PROVE side $BC >$ side AB.

Either AB is equal to, greater than, or less than BC.

If $AB = BC$, then would $c = a$. § 71

[The angles at the base of an isosceles triangle are equal.]

If $AB > BC$, then would $c > a$. § 76

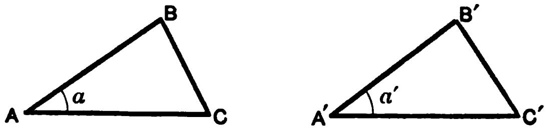
[If two sides of a triangle are unequal, the opposite angles are unequal, and the greater angle is opposite the greater side.]

But both of these conclusions contradict the hypothesis that angle $a >$ angle c.

Therefore $AB < BC$. Q. E. D.

PROPOSITION XXII. THEOREM

79. *If two triangles have two sides and the included angle of one, equal respectively to two sides and the included angle of the other, the triangles are equal.*



GIVEN—AB, AC, and a, of the triangle ABC respectively equal to $A'B'$, $A'C'$, and a', of the triangle $A'B'C'$.

TO PROVE the two triangles are equal.

Place ABC on $A'B'C'$, making AB coincide with its equal $A'B'$

Then, since $a=a'$, the side AC will fall along $A'C'$.

Also, since $AC=A'C'$, the point C will fall on C'.

Then BC will coincide with $B'C'$. Ax. *a*

[Having their extremities in the same points.]

And, since the triangles completely coincide, they are equal. § 15

Q. E. D.

80. CONSTRUCTION. *To construct an angle at a given point A' as its vertex, and on a given line $A'B'$ as a side, equal to a given angle BAC at a different vertex A.*

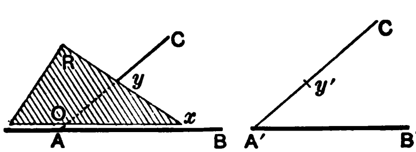


FIG. 1

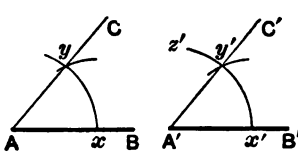


FIG. 2

First method (Fig. 1).—Place a triangular ruler, R, so that the straight edge falls along AB. Mark y on another edge where this edge cuts AC. Also mark the point A on the ruler and call it O. Draw Oy on the ruler. Then the angle BAC is reproduced on the ruler as xOy. Then, placing the ruler with O at A' and Ox along $A'B'$, retransfer the angle xOy of the ruler to the paper making $B'A'C'$. Then $B'A'C'=BAC$.

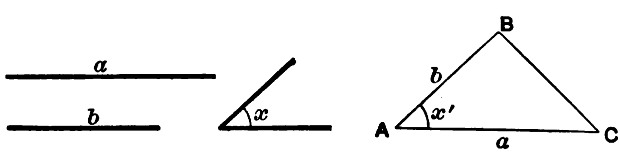
Which geometric axiom and which general axiom apply?

Evidently a visiting-card or any piece of paper with a straight edge will serve the purpose.

Second method (Fig. 2).—With A as a centre and any convenient radius describe an arc xy. With A' as a centre and the same radius describe the indefinite arc $x'z'$. Then take xy as a radius, and with x' as a centre describe an arc intersecting $x'z'$ at y'. Join $y'A'$. $y'A'B'$ is the angle required.

This cannot be proved until reaching § 89.

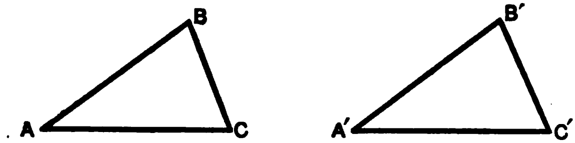
81. CONSTRUCTION. *To form a triangle with two sides and the included angle equal respectively to two lines, a and b, and a given angle, x.*



Lay off $AC=a$. Make $x'=x$ (§ 80). Lay off $AB=b$. Join BC. ABC is the triangle required, having its two sides and included angle *constructed* as required.

PROPOSITION XXIII. THEOREM

82. *If two triangles have a side and two adjacent angles of one equal to a side and two adjacent angles of the other, the two triangles are equal.*



GIVEN—in the two triangles ABC and $A'B'C'$, $AB=A'B'$, and the angles A and B equal respectively to A' and B'.

TO PROVE the triangles are equal.

Apply ABC to $A'B'C'$ making AB coincide with $A'B'$.

Then AC will fall along $A'C'$, and likewise BC along $B'C'$.

[Since the angles A and B respectively equal A' and B'.]

Hence C must fall somewhere on $A'C'$, and likewise somewhere on $B'C'$.

It must therefore fall on their intersection C'.

And, since the triangles completely coincide, they are equal. Q. E. D.

83. COR. I. *If two triangles have a side and any two angles of one equal respectively to a side and two similarly situated angles of the other, the triangles are equal.*

Hint.—Reduce to the preceding Proposition by § 60.

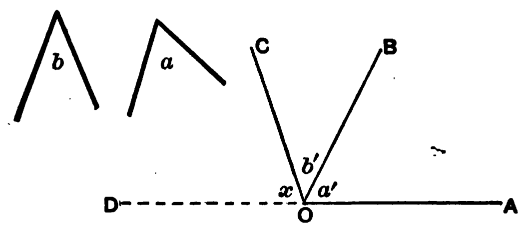
Question.—In how many ways can ABC and $A'B'C'$ have a side and two similarly situated angles equal? Draw two triangles having a side and two angles of each equal but without having the angles similarly situated.

84. *Defs.*—The **hypotenuse** of a right triangle is the side opposite the right angle. The other sides are called the **perpendicular sides.**

85. COR. II. *Two right triangles are equal, if the hypotenuse and an acute angle of one are respectively equal to the hypotenuse and an acute angle of the other.*

86. COR. III. *Two right triangles are equal, if a perpendicular side and an acute angle of one are respectively equal to a perpendicular side and the similarly situated acute angle of the other.*

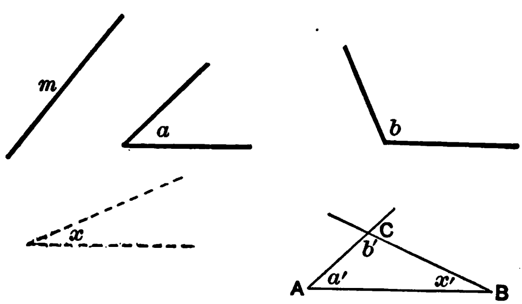
87. CONSTRUCTION. *If two angles of a triangle are equal to given angles a and b, to find the third angle.*



On any line OA construct angle $a'=a$, and on OB at the same vertex O construct $b'=b$. Produce OA to D making the angle x with OC. x is the angle required.

[Prove by § 60.]

88. CONSTRUCTION. *To form a triangle with a side and two angles equal respectively to a given line m and two angles a and b.*



Find (by § 87) x the third angle of the triangle.

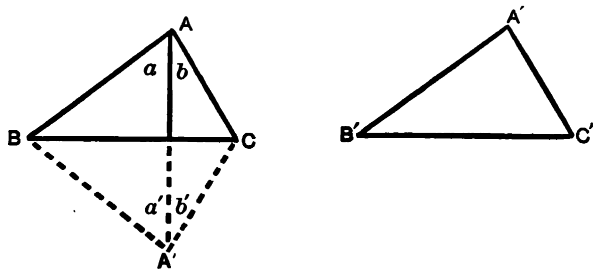
Draw any straight line AB equal to m, and at A and B construct whichever two angles of the three, a, b, x, be required to be adjacent to the given side. If the constructed sides of these angles produced meet, let C be the point of intersection. ABC is the triangle required. For AB equals m by construction, and the angles a' and b' equal a and b by construction or by proof (§ 60).

Discussion.—This problem is impossible if the two given angles are together equal to or greater than two right angles (by § 58).

Question.—Is the problem of § 81 ever impossible?

PROPOSITION XXIV. THEOREM

89. *If two triangles have their three sides respectively equal, they are equal.*



GIVEN—in the triangles ABC and $A'B'C'$, $AB = A'B'$, $BC = B'C'$, and $AC = A'C$.

TO PROVE triangle $ABC =$ triangle $A'B'C'$.

Place $A'B'C'$ so that $B'C'$ shall coincide with its equal BC, but A' shall fall on the side of BC opposite A, and join AA'.

The triangle ABA' has $AB = A'B$, that is, is isosceles. Hyp.

Hence $a = a'$. § 71

[Being base angles of an isosceles triangle.]

Likewise we may prove $b=b'$.

Adding $a+b=a'+b'$. Ax. 2

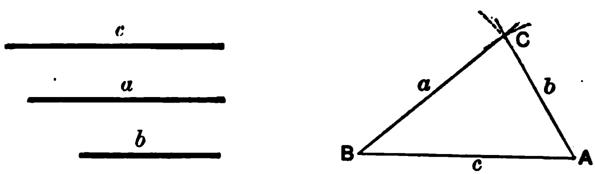
Or angle A = angle A'.

Hence triangle ABC = triangle $A'B'C'$. § 79

[Having two sides and the included angles equal.]

Q. E. D.

90. CONSTRUCTION. *To form a triangle with its three sides equal to given lines a, b, and c.*



Draw AB equal to c. From A as a centre and with b as a radius describe an arc. From B as a centre with a as a radius describe another arc. If these arcs intersect join C, their intersection, with A and B. ABC is the required triangle.

Discussion.—The problem is impossible if one of the given lines is equal to or greater than the sum of the other two.

91. *Exercise.*—By Proposition XXIV. prove that each of the following constructions is correct:

(1.) For erecting a perpendicular, as in § 21, second method.

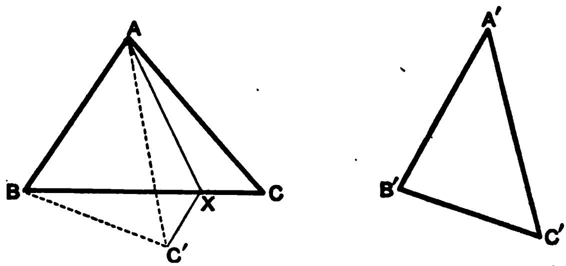
(2.) For making an angle equal to a given angle, as in § 80, second method.

Question.—If two quadrilaterals have their sides equal, each to each, are they necessarily equal?

Question.—In stating Proposition XXIV. does it matter in what order the sides are arranged?

PROPOSITION XXV. THEOREM

92. *If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first is greater than the third side of the second.*



GIVEN—two triangles ABC and $A'B'C'$ having $AB=A'B'$ and $AC=A'C'$, but angle $A>$angle A'.

TO PROVE $BC>B'C'$.

Apply $A'B'C'$ to ABC making $A'B'$ coincide with its equal AB.

The angle A' will fall within the angle BAC.

Draw AX bisecting the angle CAC' and meeting BC in X. Join $C'X$.

In the two triangles ACX and $AC'X$

	$AC=AC'$,	Hyp.
	$AX=AX$,	Iden.
	angle $CAX=$angle $C'AX$.	Cons.
Hence	triangle $ACX=$triangle $AC'X$.	§ 79
Hence	$XC=XC'$.	
Now	$BC'<BX+XC'$.	§ 7

[A straight line is the shortest path between any two of its points.]

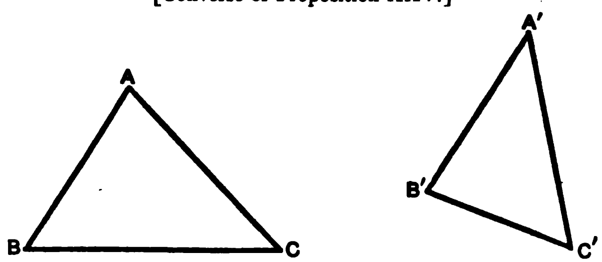
Substituting XC for its equal XC',

$$BC' < BX + XC.$$

Or $$BC' < BC.$$ Q. E. D.

PROPOSITION XXVI. THEOREM

93. ***If two triangles have two sides of one equal to two sides of the other but the third side of the first greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.*** [Converse of Proposition XXV.]



GIVEN—in the triangles ABC and $A'B'C'$, $AB = A'B'$ and $AC = A'C'$, but $BC > B'C'$.

TO PROVE angle $A >$ angle A'.

Angle A is either equal to, less than, or greater than angle A'.

If $A = A'$, then would $BC = B'C'$. § 79

[Triangles having two sides and the included angle respectively equal are equal.]

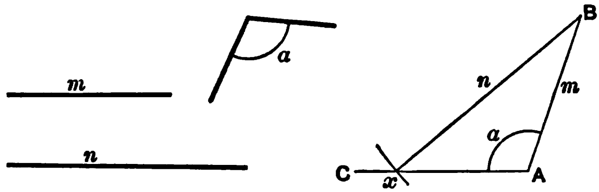
If $A < A'$ then would $BC < B'C'$. § 92

[If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first is greater than the third side of the second.]

But both these conclusions contradict the hypothesis.

Therefore $A > A'$. Q. E. D.

94. CONSTRUCTION. *To form a triangle when two sides, m and n, and an angle opposite one of them, a, are given.*

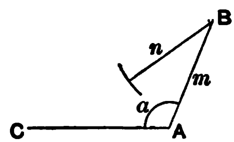


By § 80 construct the given angle a at any vertex A. On one of its sides lay off AB equal to m. From B as a centre with n as a radius draw an arc intersecting the other side at x. ABx is the triangle required.

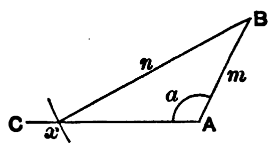
Discussion.—We may classify two groups of cases.

GROUP I.—*a being greater than an acute angle.*

CASE I.—n not longer than m.
No solution.

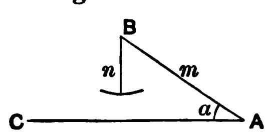


CASE II.—n longer than m.
One solution.

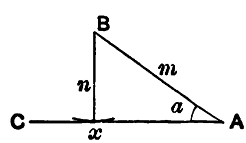


GROUP II.—*a being an acute angle.*

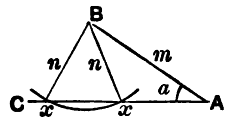
CASE I.—n shorter than the perpendicular from B to AC.
No solution.



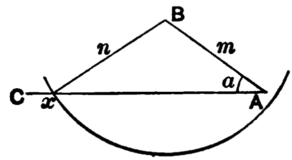
CASE II.—n equal to the perpendicular from B to AC.
One solution.



CASE III.—n longer than the perpendicular, but shorter than m. Two solutions.

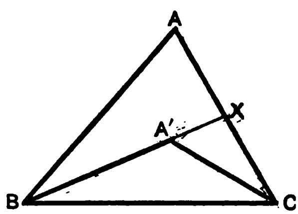


CASE IV.—n not shorter than m. One solution.



PROPOSITION XXVII. THEOREM

95. *If from a point within a triangle two straight lines are drawn to the extremities of one side, their sum will be less than the sum of the other two sides of the triangle.*



GIVEN—the triangle ABC and the lines $A'B$ and $A'C$ drawn from A' to the extremities of BC.

TO PROVE $A'B + A'C < AB + AC$.

Prolong BA' to meet AC at X.

Then $A'C < A'X + XC$. § 7

And also $A'B + A'X < XA + AB$. § 7

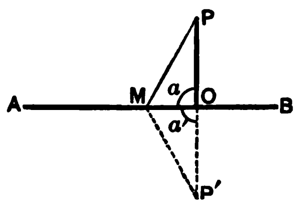
Adding, $A'C + A'B + A'X < A'X + \overline{XC + XA} + AB$. Ax. 9

Cancel $A'X$ from each side and substitute AC for $XC + XA$.

Then $A'C + A'B < AC + AB$. Q. E. D.

PROPOSITION XXVIII. THEOREM

96. *The perpendicular is the shortest line between a point and a straight line.*



GIVEN—PO the perpendicular from a point P to a straight line AB and PM *any* oblique line from P to AB.

TO PROVE $PO < PM$.

Revolve PMO about AB to form the symmetrical figure $P'MO$. § 32

Then $PO = P'O$ and $PM = P'M$.

Also PO and $P'O$ form a straight line. § 29

[If two adjacent angles (a and a') are together two right angles, their exterior sides form a straight line.]

Now $PP' < PM + MP'$. § 7

Or $2\ PO < 2\ PM$.

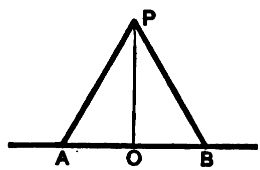
Whence $PO < PM$. Ax. 8

Q. E. D.

97. *Def.*—The "distance" from a point to a straight line means the **shortest** distance, and hence the **perpendicular** distance.

PROPOSITION XXIX. THEOREM

98. *Two oblique lines drawn from the same point in a perpendicular cutting off equal distances from the foot of the perpendicular are equal.*



GIVEN—PO perpendicular to AB, and PA and PB drawn from P cutting off $AO=BO$.

TO PROVE $PA=PB$.

In the *right* triangles POA and POB

$PO=PO$. Iden.

$AO=BO$. Hyp.

Hence triangle POA = triangle POB. § 79

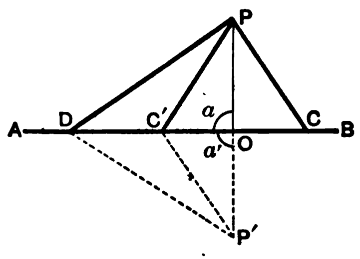
[Having two sides and included angle respectively equal.]

Therefore $PA=PB$.

[Being homologous sides of equal triangles.] Q. E. D.

PROPOSITION XXX. THEOREM

99. *Of two oblique lines drawn from the same point in a perpendicular and cutting off unequal distances from the foot, the more remote is the greater.*



GIVEN PO perpendicular to AB, and OC less than OD.

TO PROVE $PC < PD$.

Take $OC' = OC$ and join PC'.

Then $PC' = PC$. § 98

[Two oblique lines drawn from the same point in a perpendicular cutting off equal distances from the foot of the perpendicular are equal.]

Revolve the figure about AB forming the symmetrical figure $P'DO$.

Then PO and OP' form the same straight line. § 29

[If two adjacent angles (a and a') are together two right angles, their exterior sides form a straight line.]

Now $PC' + P'C' < PD + P'D$. § 95

[If from a point within a triangle, PDP', two straight lines are drawn to the extremities of one side, the sum will be less than the sum of the other two sides of the triangle.]

Substitute PC' for its equal impression $P'C'$, and likewise PD for $P'D$.

Then $2\ PC' < 2\ PD.$

Whence $PC' < PD.$ Ax. 8

Substituting PC for PC', $PC < PD.$ Q. E. D.

PROPOSITION XXXI. THEOREM

100. *If from a point in a perpendicular to a given straight line two equal oblique lines are drawn, they cut off equal distances from the foot of the perpendicular, and of two unequal oblique lines the greater cuts off the greater distance.*

[Converse of Proposition XXX.]

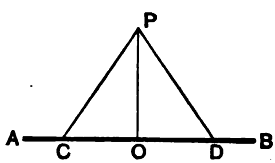


FIG. 1

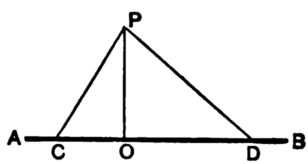


FIG. 2

I. GIVEN PO perpendicular to AB, and $PC = PD$. [Fig. 1.]

TO PROVE $OC = OD.$

OC is either greater than, less than, or equal to OD.

If $OC > OD$, then would $PC > PD$.
If $OC < OD$, then would $PC < PD$. } § 99

[Of two oblique lines drawn from the same point in a perpendicular and cutting off unequal distances from the foot of the perpendicular, the more remote is the greater.]

But both these conclusions contradict the hypothesis.

Therefore $OC = OD.$ Q. E. D.

II. GIVEN PO perpendicuiar to AB and $PD > PC$. [Fig. 2.]

TO PROVE $OD > OC.$

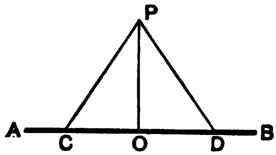


FIG. 1

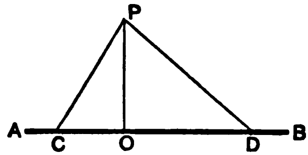


FIG. 2

OD is either equal to, less than, or greater than OC.

If $OD = OC$, then would $PD = PC$. § 98

[Two oblique lines drawn from the same point in a perpendicular cutting off equal distances from the foot of the perpendicular are equal.]

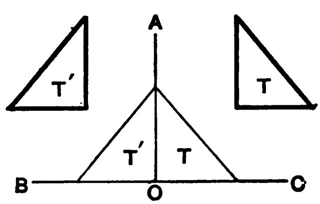
If $OD < OC$, then would $PD < PC$. § 99

[Of two oblique lines drawn from the same point in a perpendicular and cutting off unequal distances from the foot of the perpendicular, the more remote is the greater.]

But both these conclusions contradict the hypothesis.

Therefore $OD > OC$. Q. E. D.

101. COR. *Two right triangles are equal if they have the hypotenuse and a side of one equal to the hypotenuse and a side of the other.*



Hint.—Draw any two perpendicular lines, AO and BC, and place the two triangles so that their right angles shall coincide with the right angles at O and their equal sides fall along OA.

102. *Def.*—A line is the **locus** of all points which satisfy a given condition, if all points in that line satisfy the condition, and no points out of that line satisfy it.

Question—What is the locus of all points three inches from a given point? Prove it.

PROPOSITION XXXII. THEOREM

103. *The locus of all points equally distant from two given points is a straight line bisecting at right angles the line joining the given points.*

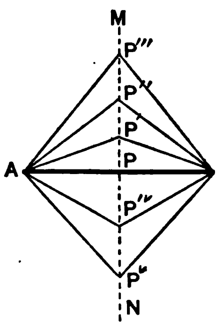


FIG. 1

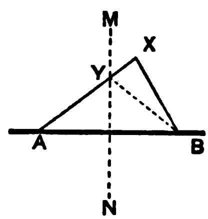


FIG. 2

GIVEN A and B, two fixed points.

TO PROVE—that the locus of all points equally distant from A and B is a straight line MN, perpendicular to AB at its middle point, P.

It is necessary to prove:

I. Every point in MN satisfies the condition of being equally distant from A and B.

II. No point without MN satisfies this condition.

I. (Fig. 1.) Draw MN perpendicular to AB at its middle point, and let P', P'', P''', P^{IV}, etc., be any points in MN.

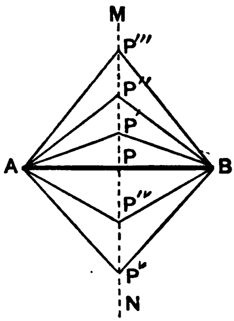


FIG. 1

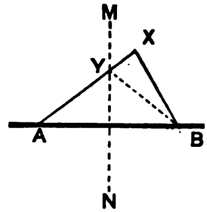


FIG. 2

Then $AP = PB$. Cons.

Hence $P'A = P'B$; $P''A = P''B$; $P'''A = P'''B$, etc. § 98

[Two oblique lines drawn from the same point in a perpendicular cutting off equal distances from the foot of the perpendicular are equal.]

That is, every point in MN is equally distant from A and B.

II. (Fig. 2.) Let X be any point without MN.

Draw XA and XB. One of these lines must cut MN in some point as Y.

Then $XB < XY + YB$. § 7

But $YA = YB$. § 98

Substituting YA for YB, $XB < XY + YA$.

Or $XB < XA$.

Hence every point without MN is unequally distant from A and B. Q. E. D.

104. COR. *Two points equally distant from the extremities of a straight line determine a perpendicular bisector to that line.*

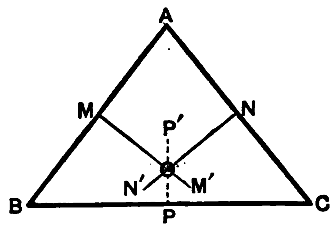
105. *Exercise.*—Show that the following methods of construction were correct:

(1.) Of dropping a perpendicular, as in § 35, second method.

(2.) Of bisecting a straight line, as in § 42, second method.

PROPOSITION XXXIII. THEOREM

106. *The three perpendicular bisectors of the sides of a triangle meet in a common point.*



GIVEN—the triangle ABC and the perpendicular bisectors MM', NN', and PP', of its sides AB, AC, and BC.

TO PROVE—MM', NN', and PP', meet in a common point.

Let O be the intersection of MM' and NN'.

O, being in MM', is equally distant from A and B. } § 103
O, being in NN', is equally distant from A and C. }

[The locus of all points equally distant from two fixed points is a straight line bisecting at right angles the line joining the fixed points.]

Hence O is equally distant from B and C.

Hence O lies in PP', the locus of points equally distant from B and C.

Therefore the three perpendicular bisectors meet in a common point. Q. E. D.

107. *Remark.*—This point is the **centre** of the triangle so far as its **vertices** are concerned—that is, it is equally distant from the vertices.

PROPOSITION XXXIV. THEOREM

108. *The bisector of an angle is the locus of all points within the angle equally distant from its sides.*

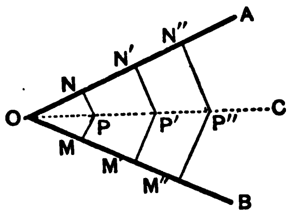


FIG. 1

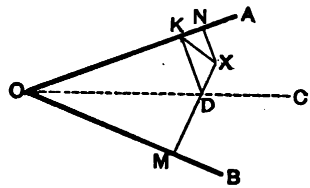


FIG. 2

GIVEN the angle AOB and its bisector OC.

TO PROVE — OC is the locus of all points equally distant from AO and BO.

It is necessary to prove:

I. That every point in OC satisfies the condition of being equally distant from AO and BO.

II. That any point without OC is unequally distant from AO and BO.

I. (Fig. 1.) Take P, any point in OC. Draw PM and PN perpendicular to OB and OA.

In the right triangles POM and PON

$OP = OP$, Iden.

angle POM = angle PON. Hyp.

Hence triangle POM = triangle PON. § 85

[Having the hypotenuse and an acute angle respectively equal.]

Therefore $PM = PN$.

[Being homologous sides of equal triangles.]

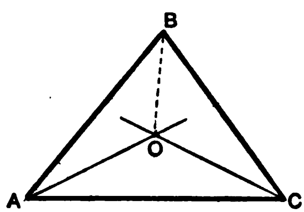
II. (Fig. 2.) Take X, any point within the angle, but not in OC. Draw XM and XN perpendicular to OB and OA.

One of these lines, as XM, must cut OC in some point, as D. Draw DK perpendicular to OA and join XK.

Then	$XN<XK$.	§ 96
And	$XK<XD+DK$.	§ 7
Hence	$XN<XD+DK$.	Ax. 13
But	$DK=DM$.	Part I
	[Since D lies in OC.]	
Therefore	$XN<XD+DM$.	
Or	$XN<XM$.	Q. E. D.

OUTLINE PROOF: $XN<XK<XD+DK=XD+DM=XM$; hence $XN<XM$.

109. COR. *The three bisectors of the angles of a triangle meet in a common point.*



Hint.—Show that the intersection of *two* of the lines must lie on the third as in Proposition XXXIII.

110. *Remark.*—This point is the **centre** of the triangle so far as its **sides** are concerned—that is, it is equally distant from the sides.

111. *Exercise.*—What is the locus of all points equally distant from two intersecting straight lines?

112. *Exercise.*—What is the locus of all points at a given distance from a fixed straight line of indefinite length?

113. *Exercise.*—What is the locus of all points at a given distance from a given line of a definite length?

PARALLELOGRAMS

114. *Defs.*—A **parallelogram** is a quadrilateral whose opposite sides are parallel.

A **rhombus** is a quadrilateral whose sides are all equal.

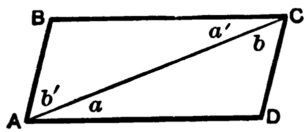
A **rectangle** is a parallelogram whose angles are all right angles.

A **square** is a rectangle whose sides are all equal.

115. *Def.*—A **diagonal** of a quadrilateral is a straight line joining opposite vertices.

PROPOSITION XXXV. THEOREM

116. *A diagonal of a parallelogram divides it into two equal triangles.*



GIVEN the parallelogram $ABCD$ and the diagonal AC.

TO PROVE—that the triangles ABC and ACD are equal.

In the triangles ABC and ACD

$AC = AC$, Iden.

$a = a'$, $b = b'$. § 48

[Being alt.-int. angles of parallel lines.]

Hence triangle ABC = triangle ACD. § 82

[Having a side and two adjacent angles in each respectively equal.]

Q. E. D.

117. COR. I. *In any parallelogram the opposite sides and angles are equal.*

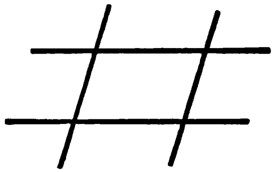


FIG. 1

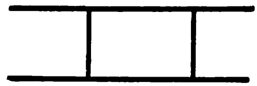


FIG. 2

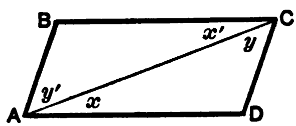
118. COR. II. *Parallels comprehended between parallels are equal.* [Fig. 1.]

119. COR. III. *Parallels are everywhere equally distant.* [Fig. 2.]

Hint.—Apply §§ 33, 36, 118.

PROPOSITION XXXVI. THEOREM

120. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*



GIVEN—any quadrilateral having its opposite sides equal, viz.: $AB=CD$, and $AD=BC$.

TO PROVE the quadrilateral is a parallelogram.

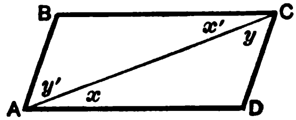
Draw the diagonal AC.

$AC=AC$. Iden.

$\left.\begin{array}{l} AB=CD. \\ AD=BC. \end{array}\right\}$ Hyp.

Hence triangle ABC = triangle ACD. § 89

[Having three sides respectively equal.]



And $x = x'$.

[Being homologous angles of equal triangles.]

Therefore BC is parallel to AD. § 43

[When two straight lines (BC and AD) are cut by a third straight line (AC) making the alternate-interior angles (x and x') equal, the straight lines are parallel.]

In like manner, using y and y', we may prove AB parallel to CD.

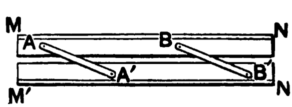
Therefore $ABCD$, having its opposite sides parallel, is a parallelogram. Q. E. D.

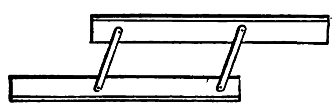
121. A "parallel ruler" is formed by two rulers (MN and $M'N'$), usually of wood pivoted to two metal strips (AA' and BB'), under the following conditions:

(1.) The distances on the rulers between pivots are equal: i. e., $AB = A'B'$.

(2.) The distances on the strips between pivots are equal; i. e., $AA' = BB'$.

(3.) In each ruler the edge is parallel to the line of pivots; i. e., AB is parallel to MN, and $A'B'$ is parallel to $M'N'$.





122. *Exercise.*—Prove: (1.) the quadrilateral whose vertices are the pivots (i. e., the figure $ABB'A'$) is always a parallelogram, whether the ruler be closed or opened.

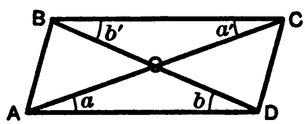
(2.) The edges of the rulers are always parallel (i. e., MN and $M'N'$ are parallel).

123. *Exercise.*—Show how to use the parallel ruler for drawing a straight line through a given point parallel to a given straight line, and prove the method correct.

Extend the method so as to apply even when the point is at a great distance from the line.

PROPOSITION XXXVII. THEOREM

124. *The diagonals of a parallelogram bisect each other.*



GIVEN—a parallelogram $ABCD$ and its diagonals AC and BD intersecting at O.

TO PROVE $AO=OC$ and $OB=OD$.

In the triangles BOC and AOD,

$a=a'$ and $b=b'$. § 48

[Being alt.-int. angles of parallel lines.]

Also $BC=AD$. § 117

[Being opposite sides of a parallelogram.]

Hence triangle BOC=triangle AOD. § 82

[Having a side and two adjacent angles respectively equal.]

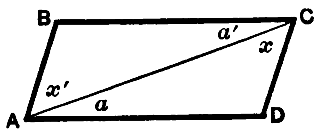
Therefore $AO=OC$ and $BO=OD$.

[Being corresponding sides of equal triangles.] Q. E. D.

125. *Exercise.*—Show that O is a centre of symmetry—that is, that if the figure be turned half way round about O as a pivot (so that OA falls along OC), it will coincide with itself.

PROPOSITION XXXVIII. THEOREM

126. *A quadrilateral which has two of its sides equal and parallel is a parallelogram.*



GIVEN—the quadrilateral $ABCD$ having BC equal and parallel to AD.

TO PROVE $ABCD$ is a parallelogram.

Draw the diagonal AC.

In the triangles ABC and ACD,

$AC = AC$, Iden.

$AD = BC$, Hyp.

angle a = angle a'. § 48

[Being alt.-int. angles.]

Therefore triangle ABC = triangle ACD. § 79

[Having two sides and the included angle respectively equal.]

Hence $x = x'$.

[Being homologous angles of equal triangles.]

Hence AB is parallel to CD. § 43

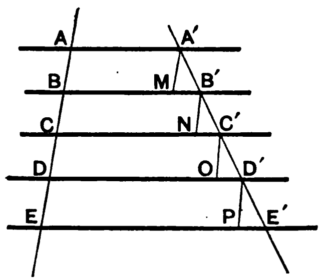
[When two straight lines are cut by a third straight line, making the alt.-int. angles equal, the lines are parallel.]

Therefore $ABCD$ is a parallelogram.

[Having its opposite sides parallel.] Q. E. D.

PROPOSITION XXXIX. THEOREM

127. *If any number of parallels intercept equal parts on one cutting line, they intercept equal parts on every other cutting line.*



GIVEN—AA', BB', CC', DD', EE', any number of parallel lines cutting off the equal parts AB, BC, CD, DE, on AE.

TO PROVE—the parts on any other line $A'E'$ are also equal, viz.: $A'B'$, $B'C'$, $C'D'$, $D'E'$.

Construct parallels to AE through the points A', B', C', D'.

Then $AB=A'M;\ BC=B'N;$ etc. § 118

[Parallels comprehended between parallels are equal.]

But $AB=BC=$etc. Hyp.

Therefore $A'M=B'N=$etc. Ax. 1

Also, in the triangles $A'MB'$, $B'NC'$, etc.,

angle $A'=$angle $B'=$etc. § 49

[Being corresponding angles of parallels.]

And angle $M=$angle $N=$etc. § 51

[Having their sides parallel and in the same order.]

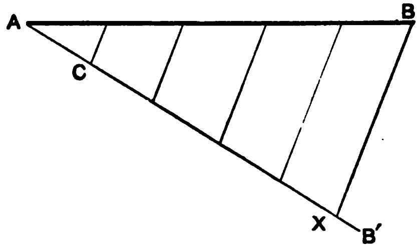
Hence triangle $A'MB'=$triangle $B'NC'=$etc. § 83

[Having a side and two angles respectively equal.]

Hence $A'B'=B'C'=C'D'=D'E'$.

[Being homologous sides of equal triangles.] Q. E. D.

128. CONSTRUCTION. *To divide a given line AB into any number of equal parts.*



From A draw any indefinite line AB' and lay off upon it any length AC.

Apply AC the required number of times on AB' and suppose X to be the last point of division. Join XB.

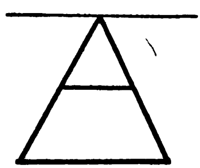
From the various points of division draw parallels to XB.

These parallels will cut AB in the required points of division.

Prove this method correct by Proposition XXXIX.

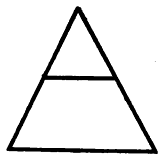
PROBLEMS

129. *Exercise.*—A straight line parallel to the base of a triangle and bisecting one side bisects the other also.



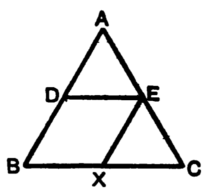
Hint.—Apply § 127.

130. *Exercise*—A straight line joining the middle points of two sides of a triangle is parallel to the third side.



Hint.—Show that this line coincides with a line drawn as in § 129.

131. *Exercise.*—A straight line joining the middle points of two sides of a triangle equals half the third side.

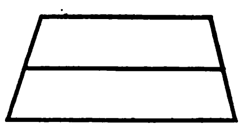


Hint.—Prove $DE=BX$, and $DE=XC$.

132. *Defs.*—A **trapezoid** is a quadrilateral, two of whose sides are parallel.

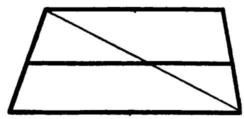
The parallel sides are called the **bases.**

133. *Exercise.*—A straight line parallel to the bases of a trapezoid and bisecting one of the remaining sides bisects the other also.



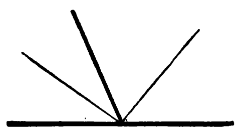
134. *Exercise.*—A straight line joining the middle points of the two non-parallel sides of a trapezoid is parallel to the bases.

135. *Exercise.*—A straight line joining the middle points of the two non-parallel sides of a trapezoid equals half the sum of the bases.



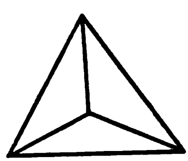
Hint.—Draw a diagonal and apply § 131.

136. *Exercise.*—The bisectors of two supplementary-adjacent angles are perpendicular.



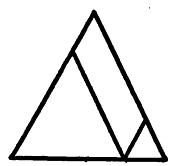
137. *Exercise.*—Any side of a triangle is greater than the difference of the other two.

138. *Exercise.*—The sum of the three lines from any point within a triangle to the three vertices is less than the sum of the three sides, but greater than half their sum.

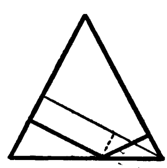


Hint.—Apply §§ 7 and 95.

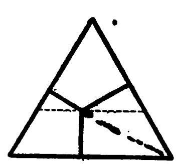
139. *Exercise.*—If from a point in the base of an isosceles triangle parallels to the sides are drawn, a parallelogram is formed, the sum of whose four sides is the same wherever the point is situated (and is equal to the sum of the equal sides).



140. *Exercise.*—If from a point in the base of an isosceles triangle perpendiculars to the sides are drawn, their sum is the same wherever the point is situated (and is equal to the perpendicular from one extremity of the base to the opposite side).

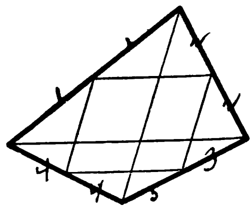


141. *Exercise.*—If from a point within an equilateral triangle perpendiculars to the three sides are drawn, the sum of these lines is the same wherever this point is situated (and is equal to the perpendicular from any vertex to the opposite side).



Hint.—Apply § 140.

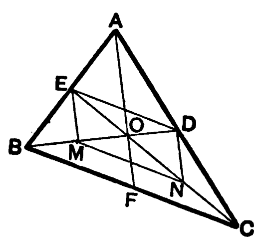
142. *Exercise.*—The straight lines joining the middle points of the adjacent sides of any quadrilateral form a parallelogram.



Hint.—Apply § 130.

143. *Def.*—A **median** of a triangle is a straight line from a vertex to the middle point of the opposite side.

144. *Exercise.*—The three medians of any triangle intersect in a common point which is two-thirds of the distance from each vertex to the middle of the opposite side.



Hint.—*Two* of these lines, *CE* and *BD*, meet at some point *O*.

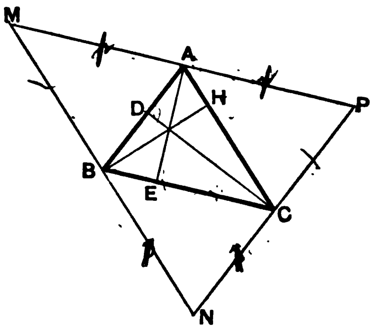
Take *M* and *N*, the middle points of *BO* and *CO*.

Draw *EDNM*. Prove it is a parallelogram by proving *ED* and *MN* each parallel to and equal to half of *BC*.

Then prove $OE=ON=NC$, and $DO=OM=MB$.

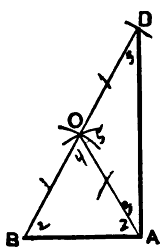
Thus we have proved that one of the medians, as *BD*, is cut by another, *CE*, at a point two-thirds of its length from *B*. We may likewise prove that it is also cut by the third median in the same point. Hence, etc.

145. *Exercise.*—The perpendiculars from the vertices of a triangle to the opposite sides meet in a point.



Hint.—Draw through each vertex a parallel to the opposite side. Prove *AE*, *BH*, and *CD* are perpendicular bisectors of the sides of the new triangle *MNP*, and apply § 106.

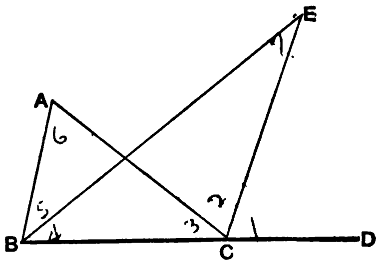
146. *Exercise.*—Prove that the following is a correct method for erecting a perpendicular from a point *A* in a line *AB*.



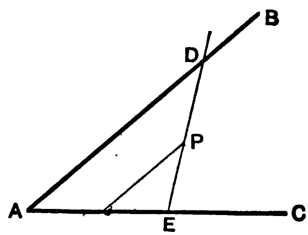
With *A* as a centre describe an arc. With the same radius and any other point, *B*, in the line as a centre, describe a second arc intersecting the first at *O*. With *O* as a centre and the same radius describe [illegible] arc. Join *BO* and produce to meet the third arc at *D*. [illegible] perpendicular required.

[illegible] four right angles of the two triangles, two are at *O*. Show that [illegible] remainder are at *A*.

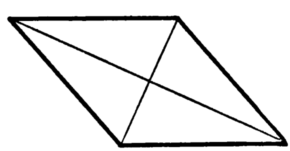
147. *Exercise.*—Given ABC, any triangle. Produce BC. Draw CE bisecting angle ACD, and BE bisecting angle ABC. Prove angle E equals half of angle A.



148. *Exercise.*—Given any angle A and any point P within it. Show a method of drawing a line through P to the sides of the angle which shall be bisected at P.



149. *Exercise.*—The diagonals of a rhombus bisect each other at right angles, and also bisect the angles of the rhombus.



is cut by anothe
may likewise pro
Hence, etc.

PLANE GEOMETRY

BOOK II

THE CIRCLE

150.* *Def.*—A **circle** is a plane figure bounded by a line, all points of which are equally distant from a point within called the **centre.**

151.* *Defs.*—The line which bounds the circle is called its **circumference.**

An **arc** is any part of a circumference.

152.* *Def.*—Any straight line from the centre to the circumference is a **radius.**

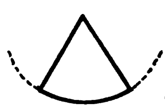
By the definition of a circle all its radii are equal.

153. *Def.*—A **chord** is a straight line having its extremities in the circumference.

154. *Def.*—A **diameter** is a chord through the centre.

All diameters are equal, each being twice a radius.

155. *Defs.*—A **sector** is that portion of a circle bounded by two radii and the intercepted arc.



SECTOR

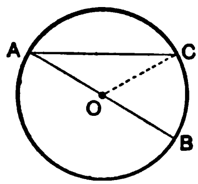
The angle between the radii is called the **angle of the sector.**

* These definitions are repeated from § 20.

156. *Def.*—**Concentric** circles are circles which have the same centre.

PROPOSITION I. THEOREM

157. *The diameter of a circle is greater than any other chord.*



GIVEN—the circle ABC and AC, any chord not a diameter.

TO PROVE $AC <$ diameter AB.

Draw the radius OC.

$AC < AO + OC$. § 7

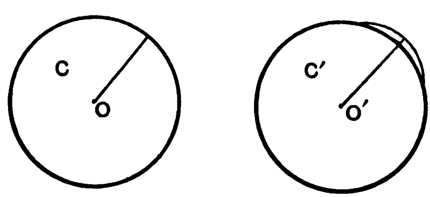
Substitute for OC the equal radius OB.

Then $AC < AO + OB$.

That is $AC < AB$. Q. E. D.

PROPOSITION II. THEOREM

158. *Circles which have equal radii are equal, and if their centres be made to coincide they will coincide throughout; conversely, equal circles have equal radii.*



I. GIVEN—any two circles, C and C' with centres O and O' and equal radii.

TO PROVE the circles C and C' are equal.

Place the circles so that O falls on O'.

Then the circumference of C will coincide with the circumference of C':

For, if any portion of one fell without the other, its distance from the centre would be greater than the distance of the other. Hence the radii would be unequal, which is contrary to the hypothesis. Ax. 10

Therefore, the circumferences coincide, and the circles coincide and are equal. Q. E. D.

II. CONVERSELY:

GIVEN two equal circles.

TO PROVE their radii equal.

Since the circles are equal they can be made to coincide, and therefore their radii will coincide, and are equal. Q. E. D.

159. COR. I. Hence, *if a circle be turned about its centre as a pivot, its circumference will continue to occupy the same position.*

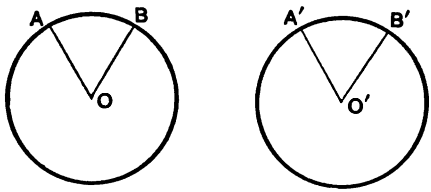
160. COR. II. *The diameter of a circle bisects the circle and the circumference.*

Hint.—Fold over on the diameter as an axis.

161. *Defs.*—The halves into which a diameter divides a circle are called **semicircles**, and the halves into which it divides the circumference are called **semicircumferences.**

PROPOSITION III. THEOREM

162. *In the same circle or equal circles, equal angles at the centre intercept equal arcs; conversely, equal arcs are intercepted by equal angles at the centre.*



I. GIVEN—equal circles and equal angles at their centres, O and O'.

TO PROVE arc AB=arc $A'B'$.

Apply the circles making the angle O coincide with angle O'.

A will coincide with A', and B with B'. § 158

[For $AO=A'O'$, and $OB=O'B'$, being radii of equal circles.]

Then the arc AB will coincide with the arc $A'B'$, and is equal to it. § 150

Q. E. D.

II. CONVERSELY:

GIVEN—equal circles having equal arcs AB and $A'B'$.

TO PROVE—the subtended angles O and O' equal.

Apply the circles making the arc AB coincide with its equal $A'B'$.

Then AO will coincide with $A'O'$, and BO with $B'O'$. Ax. *a*

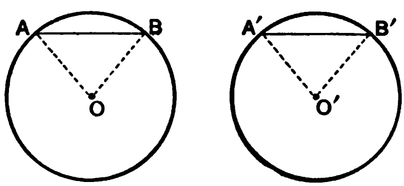
Therefore angles O and O' coincide and are equal. Q. E. D.

163. *Exercise.*—In the same circle or equal circles equal angles at the centre include equal sectors, and conversely.

The proof is analogous to the preceding, requiring "sector" in place of "arc."

PROPOSITION IV. THEOREM

164. *In the same circle or equal circles, equal chords subtend equal arcs; conversely, equal arcs are subtended by equal chords.*



GIVEN—equal circles, O and O', and equal chords, AB and $A'B'$.

TO PROVE arc AB = arc $A'B'$.

Draw the four radii OA, OB, $O'A'$, $O'B'$.

In the triangles AOB and $A'O'B'$

$AB = A'B'$. Hyp.

$AO = A'O'$, and $OB = O'B'$. § 158

[Being radii of equal circles.]

Hence triangle AOB = triangle $A'O'B'$. § 89

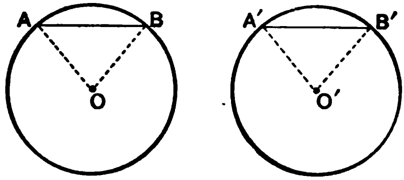
[Having three sides respectively equal.]

Hence angle O = angle O'.

[Being corresponding angles of equal triangles.]

Therefore arc AB = arc $A'B'$. § 162

Q. E. D.



CONVERSELY:

GIVEN—equal circles O and O', and arc AB = arc $A'B'$.

TO PROVE chord AB = chord $A'B'$.

Since the arcs are equal, angle O = angle O'. § 162

And the four radii are equal. § 158

Hence triangle AOB = triangle $A'O'B'$. § 79

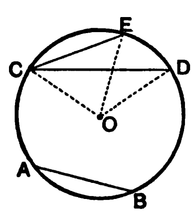
[Having two sides and the included angle respectively equal.]

Therefore chord AB = chord $A'B'$.

[Being corresponding sides of equal triangles.] Q. E. D.

PROPOSITION V. THEOREM

165. *In the same circle or equal circles, if two arcs are unequal and each is less than half a circumference, the greater arc is subtended by the greater chord; conversely, the greater chord subtends the greater arc.*



GIVEN arc CD greater than arc AB.

TO PROVE chord CD greater than chord AB.

Construct upon the greater arc CD an arc CE equal to arc AB.

Then chord CE=chord AB. § 164

Draw the radii OC, OD, OE.

Angle COE is less than angle DOC, being included within it. Ax. 10

Then triangles COE and DOC have two sides (the radii) respectively equal, but the included angles unequal.

Therefore chord CE<chord CD. § 92

Substituting AB for CE,

chord AB<chord CD. Q. E. D.

CONVERSELY:

GIVEN chord CD greater than chord AB.

TO PROVE arc CD greater than arc AB.

As before, construct arc CE equal to arc AB.

Then chord CE=chord AB. § 164

But chord CD>chord AB. Hyp.

Substituting CE for AB,

chord CD>chord CE.

Then the triangles COE and DOC have two sides respectively equal, but the third sides unequal.

Therefore angle COE<angle COD. § 93

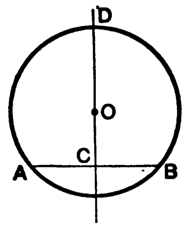
Hence OE, being within the angle DOC, must cut off the arc CE less than the arc CD.

Substituting arc AB for arc CE,

arc AB<arc CD. Q. E. D.

PROPOSITION VI. THEOREM

166. *The perpendicular bisector of a chord passes through the centre of the circle.*



GIVEN—circle OAB, chord AB, and CD, the perpendicular bisector of AB.

TO PROVE that CD passes through the centre O.

CD contains all points equally distant from A and B. § 103
[Being the locus of such points.]

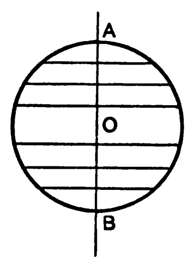
But O is such a point, being the centre.

Therefore CD contains O. Q. E. D.

167. COR. *The diameter perpendicular to a chord bisects it and the subtended arc.*

Hint.—Prove this diameter coincides with the perpendicular bisector. Then draw radii OA and OB, and apply § 162.

168. *Exercise.*—The locus of the middle points of all chords parallel to a given straight line is a diameter perpendicular to the chords.



The student is cautioned in this, and in exercises about loci in general, not to regard the locus found and proved until he has shown *two* things:

(1.) That every point in the proposed locus satisfies the proposed condition, i. e., is the middle point of one of the parallel chords.

(2.) That every point outside of the proposed locus does not satisfy the required condition, i. e., is not the middle point of any of the parallel chords.

Thus the radius is not the locus, being too small (i. e., requirement 1 would be fulfilled, but not 2); and the diameter produced is not, being too large (i. e., requirement 2 would be fulfilled, but not 1).

Some exercises on loci are more easily proved by showing:

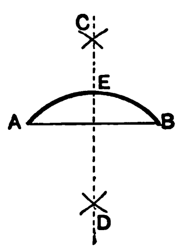
(1.) That every point in the proposed locus satisfies the proposed conditions.

(2.) That every point that satisfies the proposed conditions is in the proposed locus.

The student should show that this method of establishing a locus is equivalent to the previous method.

He may also prove by this method §§ 103 and 108.

169. CONSTRUCTION. *To bisect a given arc.*



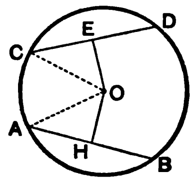
GIVEN	the arc AEB.
TO CONSTRUCT	its bisector.

From A and B as centres, with equal radii greater than a half of AB, describe arcs intersecting at C and D. Draw CD. This line bisects the arc at E.

Hint.—For proof apply § 167.

PROPOSITION VII. THEOREM

170. *In the same circle or equal circles, equal chords are equally distant from the centre; conversely, chords equally distant from the centre are equal.*



GIVEN CD and AB, equal chords.

TO PROVE—they are at equal distances, EO and HO, from the centre.

Construct radii OC and OA.

E and H are the middle points of CD and AB. § 167

In the right triangles OCE and OAH

$CE = AH$, being halves of equals. Ax. 8

$OC = OA$, being radii.

Hence the triangles are equal. § 101

[Having a side and hypotenuse respectively equal.]

Therefore $OE = OH$. Q. E. D.

CONVERSELY:

GIVEN $OE = OH$.

TO PROVE $CD = AB$.

In the right triangles OCE and OAH

$OE = OH$. Hyp.

$OC = OA$, being radii.

Hence the triangles are equal. § 101

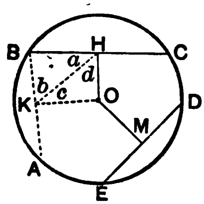
Therefore $CE = AH$.

And $CD = AB$, being doubles of equals. Ax. 7

Q. E. D.

PROPOSITION VIII. THEOREM

171. *In the same circle or equal circles, the less of two chords is at the greater distance from the centre; conversely, the chord at the greater distance from the centre is the less.*



GIVEN chord $ED <$ chord BC.

TO PROVE distance $OM >$ distance OH.

Construct from B chord $BA = ED$.

Then its distance $OK = OM$.

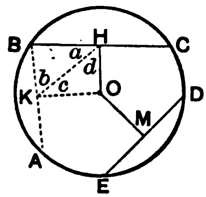
And $AB < BC$.

Join KH.

K and H are the middle points of AB and BC. § 167

Hence $BK < BH$. Ax. 8

[Being halves of unequals.]



Hence angle $a <$ angle b. § 76

[Being opposite unequal sides.]

Subtracting the unequal angles from the equal right angles at H and K,

angle $d >$ angle c. Ax. 6

Therefore $OK > OH$. § 78

[Being opposite unequal angles.]

Substituting OM for OK,

$OM > OH$. Q. E. D.

SUMMARY: $ED < BC$; $BA < BC$; $BK < BH$; $a < b$; $d > c$; $OK > OH$; $OM > OH$.

CONVERSELY:

GIVEN $OM > OH$.

TO PROVE $ED < BC$.

The proof is left to the student.

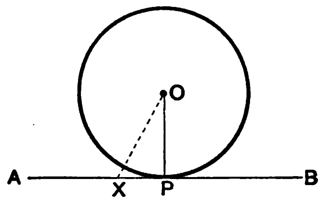
SUMMARY: $OM > OH$; $OK > OH$; $d > c$; $a < b$; $BK < BH$; $BA < BC$; $ED < BC$.

172. *Defs.*—A straight line is **tangent** to a circle if, however far produced, it meets its circumference in but one point.

This point is called the **point of tangency.**

PROPOSITION IX. THEOREM

173. *A straight line perpendicular to a radius at its extremity is tangent to the circle; conversely, the tangent at the extremity of a radius is perpendicular to that radius.*



GIVEN—AB perpendicular to the radius OP at its extremity P.

TO PROVE AB is tangent to the circle.

The perpendicular OP is less than any other line OX from O to AB. § 96

[Being the shortest distance from a point to a line.]

Hence, OX being greater than a radius, X lies without the circumference, and P is the only point in AB on the circumference. Therefore AB is tangent to the circle. Q. E. D.

CONVERSELY:

GIVEN AB tangent to the circle at P.

TO PROVE OP perpendicular to AB.

Since AB is touched only at P, any other point in AB, as X, lies without the circumference.

Hence OX is greater than a radius OP.

Therefore OP, being shorter than any other line from O to AB, is perpendicular to AB. § 96

Q. E. D.

174. COR. *A perpendicular to a tangent at the point of tangency passes through the centre of the circle.*

Hint.—Suppose a radius to be drawn to the point of tangency.

175. CONSTRUCTION. *At a point P in the circumference of a circle to draw a tangent to the circle.*

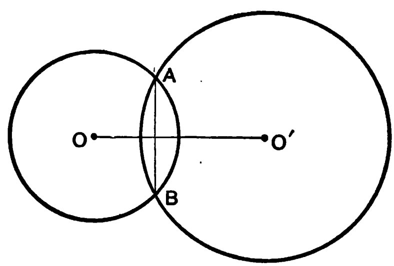
Draw the radius OP, and erect PB perpendicular to this radius at P. By § 173 PB is the tangent required.

176. *Exercise.*—The two tangents to a circle from an exterior point are equal.

Hint.—Join the given point and the centre; draw radii to points of tangency.

PROPOSITION X. THEOREM

177. *If two circumferences intersect, the straight line joining their centres bisects their common chord at right angles.*



GIVEN two circumferences intersecting at A and B.

TO PROVE—OO' joining their centres is perpendicular to AB at its middle point.

O and O' are each equally distant from A and B. § 150

Therefore OO' bisects AB at right angles. § 104

[Two points equally distant from the extremities of a straight line determine its perpendicular bisector.] Q. E. D.

MEASUREMENT

178. *Def.*—The **ratio** of one quantity to another of the same kind is the number of times the first contains the second.

Thus the ratio of a yard to a foot is three (3), or more fully $\frac{3}{1}$.

179. *Defs.*—To **measure** a quantity is to find its ratio to another quantity of the same kind. The second quantity is called the **unit of measure**; the ratio is called the **numerical measure** of the first quantity.

Thus we measure the length of a rope by finding the number of metres in it; if it contains 6 metres, we say the *numerical measure* of its length is 6, the metre being the *unit of measure.*

180. *Remark.*—If the length of one rope is 20 metres, and the length of another 5 metres, the ratio of their lengths is the number of times 5 metres is contained in 20 metres—that is, the number of times 5 is contained in 20, which is written $\frac{20}{5}$. We may accordingly restate § 178 as follows:

The ratio of two quantities of the same kind is the ratio of their numerical measures.

181. *Defs.*—Two quantities are **commensurable** if there exists a third quantity which is contained a whole number of times in each.

The third quantity is called the **common measure.**

Thus a yard and a mile are commensurable, each containing a foot a whole number of times, the one 3 times, the other 5280 times. Again, a yard and a rod are commensurable. The common measure is not, however, a foot, as a rod contains a foot $16\frac{1}{2}$ times, which is not a whole number of times. But an inch is a common measure, since the yard contains it 36 times and the rod 198 times.

182. *Def.*—Two quantities are **incommensurable** if no third quantity exists which is contained a whole number of times in each.

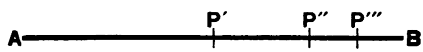
Thus it can be proved that the circumference and diameter of a circle are incommensurable; also the side and diagonal of a square.

LIMITS

183. *Def.*—A **constant** quantity is one that maintains the same value throughout the same discussion.

184. *Def.*—A **variable** is a quantity which has different successive values during the same discussion.

185. *Def.*—The **limit** of a variable is a constant *from* which the variable can be made to differ by less than any assigned quantity, but *to* which it can never be made equal.



Thus suppose a point P to move over a line from A to B in such a way that in the first second it passes over half the distance, in the next second half the remaining distance, in the third half the new remainder, and so on.

The variable is the *distance* from A to the moving-point. Its successive values are AP', AP'', AP''', etc. If the length of AB is two inches, the value of the variable is first 1 inch, then $1\frac{1}{2}$, $1\frac{3}{4}$, $1\frac{7}{8}$, etc.

(1.) P will *never* reach B, for there is always half of *some* distance remaining.

(2.) P will approach nearer to B than any quantity we may assign.

Suppose we assign $\frac{1}{1000}$ of an inch. By continually bisecting the remainder we can reduce it to less than $\frac{1}{1000}$ of an inch. Hence the distance from P to A is a variable whose limit is AB, and the distance from P to B is a variable whose limit is zero.

186. THEOREM. *If two variables approaching limits are always equal, their limits are also equal.*

For two variables that are always equal may be considered as but one variable, and must therefore approach the same limit. Q. E. D.

187. LEMMA. *If a variable x can be made smaller than any assigned quantity, then kx, the product of that variable by any constant k, can also be made smaller than any assigned quantity.*

Suppose we assign a quantity s, no matter how small.

We then choose x, so that $x < \frac{s}{k}$.

Therefore, multiplying, $kx < s$. Ax. 7

Q. E. D.

188. COR. *If a variable x can be made as small as we please, so also can x divided by any constant k.*

For $\frac{x}{k}$ is simply $\left(\frac{1}{k}\right) x$, or the product of x by a constant, which we have just proved can be made as small as we please.

189. THEOREM. *The limit of the product of a constant by a variable is the product of that constant by the limit of the variable.*

GIVEN a variable v approaching a limit V.

TO PROVE—the variable kv approaches the limit kV, where k is any constant.

I. kv can never quite equal kV.

For if $kv = kV$,

then would $v = V$, Ax. 8

which is impossible, since v approaches V as a *limit*.

II. kv can be made to differ from kV by less than any assigned quantity.

For their difference, $kV - kv$, may be written $k(V - v)$.

But $V - v$ can be made as small as we please.

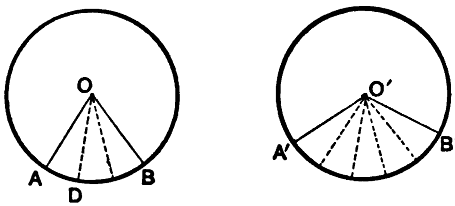
Therefore $k(V - v)$ can be made as small as we please. § 187

Therefore by definition kV is the limit of kv. Q. E. D.

190. COR. *The limit of $\frac{v}{k}$, the quotient of a variable divided by a constant, is $\frac{V}{k}$, the quotient of the limit of the variable divided by the constant k.*

PROPOSITION XI. THEOREM

191. *In the same circle or equal circles, two angles at the centre have the same ratio as their intercepted arcs.*



GIVEN the two equal circles with angles O and O'.

TO PROVE $\dfrac{\text{angle } O'}{\text{angle } O} = \dfrac{\text{arc } A'B'}{\text{arc } AB}$.

CASE I. *When the arcs are commensurable.*

Suppose AD is the common measure of the arcs, and is contained three times in AB and five times in $A'B'$.

Then $\dfrac{\text{arc } A'B'}{\text{arc } AB} = \dfrac{5}{3}$. § 180

Draw radii to the several points of division.

The angles O and O' will be subdivided into 3 and 5 parts, *all equal.* § 162

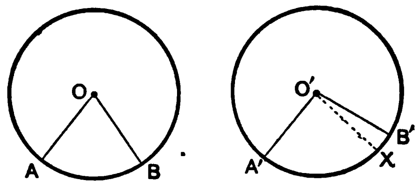
[Being subtended by equal arcs in the same or equal circles.]

Hence $\dfrac{\text{angle } O'}{\text{angle } O} = \dfrac{5}{3}$. § 180

Comparing, $\dfrac{\text{angle } O'}{\text{angle } O} = \dfrac{\text{arc } A'B'}{\text{arc } AB}$. Ax. 1

Q. E. D.

CASE II. *When the arcs are incommensurable.*



Suppose AB to be divided into any number of equal parts and apply one of these parts to $A'B'$ as a measure as often as it will go.

Since AB and $A'B'$ are incommensurable, there will be a remainder XB' less than one of these parts. § 182

Since AB and $A'X$ are constructed commensurable,

$$\frac{\text{angle } A'O'X}{\text{angle } AOB} = \frac{\text{arc } A'X}{\text{arc } AB}. \qquad \text{Case I}$$

Now suppose the number of parts into which AB is divided to be indefinitely increased.

We can thus make each part as small as we please.

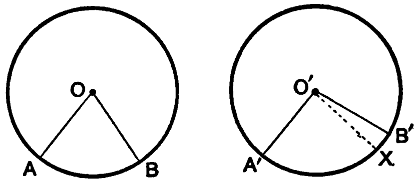
But the remainder, the arc XB', will always be less than one of these parts.

Therefore we can make the arc XB' less than any assigned quantity, though never zero.

Likewise we can make the angle $XO'B'$ less than any assigned quantity, though never zero.

Therefore $A'X$ approaches $A'B'$ as a limit.

Hence $\dfrac{A'X}{AB}$ approaches $\dfrac{A'B'}{AB}$ as a limit. § 190



Also angle $A'O'X$ approaches angle $A'O'B'$ as a limit.

Hence $\dfrac{\text{angle } A'O'X}{\text{angle } AOB}$ approaches $\dfrac{\text{angle } A'O'B'}{\text{angle } AOB}$ as a limit. § 190

Since the variables $\dfrac{A'X}{AB}$ and $\dfrac{\text{angle } A'O'X}{\text{angle } AOB}$ are always equal, so also are their limits.

That is, $$\frac{A'B'}{AB}=\frac{\text{angle } A'O'B'}{\text{angle } AOB}.$$ § 186

Q. E. D.

192. *Exercise.*—In the same circle or equal circles, two sectors have the same ratio as their angles.

The proof is analogous to the preceding, requiring "sector" in place of "arc."

193. *Remark.*—The preceding proposition is often expressed thus:

An angle at the centre *is measured by* its intercepted arc.

This means simply that if the angle is doubled, the intercepted arc will be doubled; if the angle is halved, the intercepted arc will be halved; if the angle is tripled, the intercepted arc will be tripled; and, in general, if the angle is increased or diminished in any ratio, the intercepted arc will be increased or diminished in the same ratio.

194. *Defs.*—A **degree of angle** is one-ninetieth of a right angle.

A **degree of arc** is the arc intercepted by a degree of angle at the centre.

The arc intercepted by a right angle at the centre is called a **quadrant.**

Hence a quadrant contains 90 degrees (90°) of arc, since a right angle contains 90° of angle.

Also, since four right angles contain 360° of angle, and four right angles intercept a complete circumference, a circumference contains 360° of arc.

Hence a quadrant is one-quarter of a circumference.

195. *Remark.*—These definitions suggest a special form of stating Proposition XI., viz.: The ratio of any angle at the centre to a degree of angle equals the ratio of the intercepted arc to the degree of arc, or more briefly: *An angle at the centre contains as many degrees of angle as its intercepted arc contains degrees of arc;* or still again, the numerical measure of an angle at the centre equals the numerical measure of its intercepted arc, the unit of angle being a degree of angle, and the unit of arc being a degree of arc.

The student will be tempted to still further simplify the statement by saying "an angle at the centre *equals* its intercepted arc." This, however, is erroneous, because an angle and an arc are not quantities of the same kind, and can no more be called equal than 23 pounds can be said to be equal to 23 yards.

196. *Def.*—An angle is said to be **inscribed** in a circle, if its vertex lies in the circumference and its sides are chords.

PROPOSITION XII. THEOREM

197. *An inscribed angle is measured by one-half its intercepted arc.**

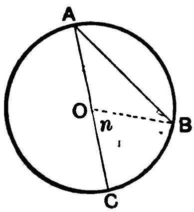


FIG. 1

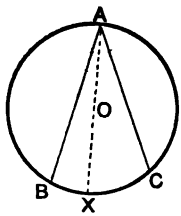


FIG. 2

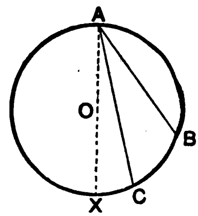


FIG. 3

GIVEN the inscribed angle BAC.

TO PROVE—angle BAC is measured by one-half of arc BC.

CASE I. *When one side AC of the angle is a diameter* (Fig. 1).

Draw the radius OB.

$OA = OB$. § 152

[Being radii.]

Hence angle $A =$ angle B. § 71

[Being base angles of an isosceles triangle.]

But angle $n =$ angle $A +$ angle B. § 59

[The exterior angle of a triangle equals the sum of the two opposite interior angles.]

Substituting A for B, $n = 2A$.

But n is measured by arc BC. § 193

Hence half of n, or A, is measured by $\frac{1}{2}$ arc BC. Q. E. D.

* This proposition is first found proved in Euclid (about 300 B.C.), though at least one case, viz., Cor. II. was stated earlier by Thales (about 600 B.C.), the founder of Greek mathematics and philosophy.

CASE II. *When the centre O is within the angle* (Fig. 2).

Construct the diameter AX.

Angle XAC is measured by $\frac{1}{2}$ arc XC. Case I

Angle XAB is measured by $\frac{1}{2}$ arc XB. Case I

Adding, angle BAC is measured by $\frac{1}{2}$ arc $XC + \frac{1}{2}$ arc XB. Ax. 2

Or by $\frac{1}{2}$(arc $XC +$ arc XB).

That is by $\frac{1}{2}$ arc BC.

CASE III. *When the centre is without the angle* (Fig. 3).

Construct the diameter AX.

Angle XAB is measured by $\frac{1}{2}$ arc XB. Case I

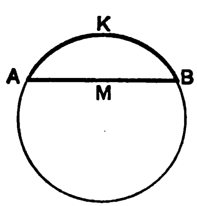
Angle XAC is measured by $\frac{1}{2}$ arc XC. Case I

Subtracting, angle BAC is measured by $\frac{1}{2}$ arc BC. Ax. 3

Q. E. D.

198. *Exercise.*—If the inscribed angle is 37° of angle, how many degrees of arc are there in the intercepted arc? How many in the remainder of the circumference? If the intercepted arc is 17°, how large is the inscribed angle?

199. *Defs.*—A **segment** of a circle is the portion of a circle included between an arc and its chord, as $AKBM$.



200. *Def.*—An angle is **inscribed** in a segment of a circle when its vertex is in the arc of the segment and its sides pass through the extremities of that arc.

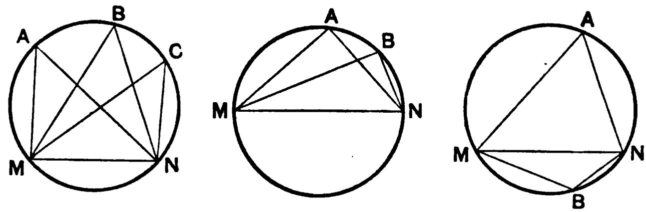


FIG. 1 FIG. 2 FIG. 3

201. COR. I. *All angles* (A, B, C, Fig. 1) *inscribed in the same segment are equal.*

For they are measured by one-half the same arc MN.

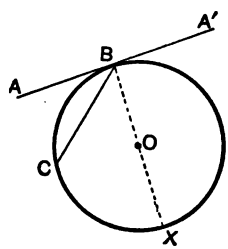
202. COR. II. *An angle* (A, B, Fig. 2) *inscribed in a semicircle is a right angle.*

203. COR. III. *An angle* (A, Fig. 3) *inscribed in a segment greater than a semicircle is an acute angle.*

204. COR. IV. *An angle* (B, Fig. 3) *inscribed in a segment less than a semicircle is an obtuse angle.*

PROPOSITION XIII. THEOREM

205. *An angle formed by a tangent and a chord is measured by one-half its intercepted arc.*



GIVEN—the angle ABC formed by the tangent AB and the chord BC.

TO PROVE—angle ABC is measured by one-half the arc BC.

Construct the diameter BX.

Since a right angle is measured by one-half a semicircumference,

angle ABX is measured by $\frac{1}{2}$ arc BCX.

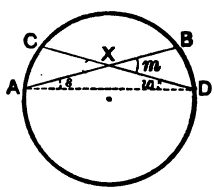
But angle CBX is measured by $\frac{1}{2}$ arc CX. § 197

Subtracting, angle ABC is measured by $\frac{1}{2}$ arc BC. Q. E. D.

206. *Exercise.*—An arc contains 16°; at its extremities tangents are drawn. What kind of a triangle do they form with the chord, and how large is each angle?

PROPOSITION XIV. THEOREM

207. *The angle between two chords which intersect within the circumference is measured by one-half the sum of its intercepted arc and the arc intercepted by its vertical angle.*



GIVEN two intersecting chords AB and CD.

TO PROVE—angle BXD is measured by one-half the sum of the arcs BD and AC.

Join AD.

Now $m = s + w$. § 59

[An exterior angle of a triangle equals the sum of the opposite interior angles.]

But angle s is measured by $\frac{1}{2}$ arc BD. § 197

And angle w is measured by $\frac{1}{2}$ arc AC. § 197

Hence m is measured by $\frac{1}{2}$ (arc $BD + \frac{1}{2}$ arc AC). Ax. 2

Q. E. D.

208. *Exercise.*—One angle of two intersecting chords subtends 30° of arc; its vertical angle subtends 40°. How large is the angle? If an angle of two intersecting chords is 15°, and its intercepted arc is 20°, how large is the opposite arc?

209. *Def.*—A **secant** of a circle is a straight line which cuts the circle.

It is therefore a chord produced.

PROPOSITION XV. THEOREM

210. *The angle between two secants intersecting without the circumference, the angle between a tangent and a secant, and the angle between two tangents, are each measured by one-half the difference of the intercepted arcs.*

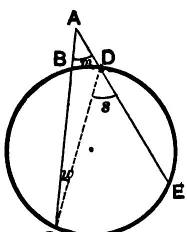


FIG. 1

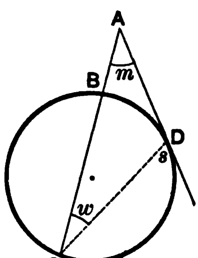


FIG. 2

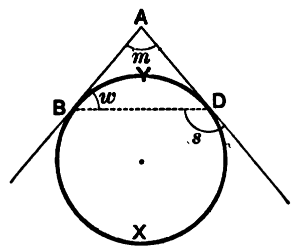


FIG. 3

CASE I. *Two secants* (Fig. 1).

GIVEN two secants, AC and AE.

TO PROVE—angle m is measured by $\frac{1}{2}$ (arc CE—arc BD).

Join CD.

Then $m+w=s$. § 59

[An exterior angle of a triangle is equal to the sum of the two opposite interior angles.]

Hence	$m=s-w$.	Ax. 3
But	s is measured by $\frac{1}{2}$ arc CE.	§ 197
And	w is measured by $\frac{1}{2}$ arc BD.	§ 197
Hence	m is measured by $\frac{1}{2}$ (arc CE — arc BD).	Ax. 3

Q. E. D.

CASE II. *A tangent and a secant* (Fig. 2).

GIVEN tangent AD and secant AC.

TO PROVE m is measured by $\frac{1}{2}$ (arc DC — arc BD).

Join CD.

	$m=s-w$.	§ 59
	s is measured by $\frac{1}{2}$ arc DC.	§ 205
	w is measured by $\frac{1}{2}$ arc BD.	§ 197
Hence	m is measured by $\frac{1}{2}$ (arc DC — arc BD).	Ax. 3

Q. E. D.

CASE III. *Two tangents* (Fig. 3).

	$m=s-w$.	§ 59
	s is measured by $\frac{1}{2}$ arc BXD.	§ 205
	w is measured by $\frac{1}{2}$ arc BYD.	§ 205
Hence	m is measured by $\frac{1}{2}$ (arc BXD — arc BYD).	Ax. 3

Q. E. D.

211. *Exercises.*—In Fig. 1, if CE is 50° and BD is 10°, what is m?

In Fig. 1, if m is 16° and BD is 15°, what is CE?

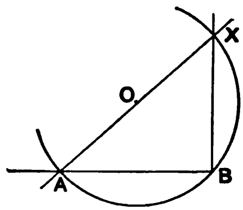
In Fig. 2, if m is 31° and arc DC is 150°, what is arc BD? and what is arc BC?

In Fig. 3, if arc BD is 47°, what is BXD, and what is m?

In Fig. 3, if m is 33°, what are the arcs BXD and BYD?

212. CONSTRUCTION. *At a given point in a straight line to erect a perpendicular.*

[Three methods have been already given, §§ 21, 146.]



GIVEN the straight line AB.

TO CONSTRUCT a perpendicular to AB at B.

With any convenient point O as a centre, and OB as a radius, describe a circumference cutting AB at A and B.

Join OA and produce to meet the circumference at X.

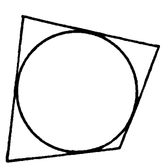
BX is the perpendicular required.

Proof.—Angle ABX is inscribed in a semicircle, and therefore a right angle. § 202

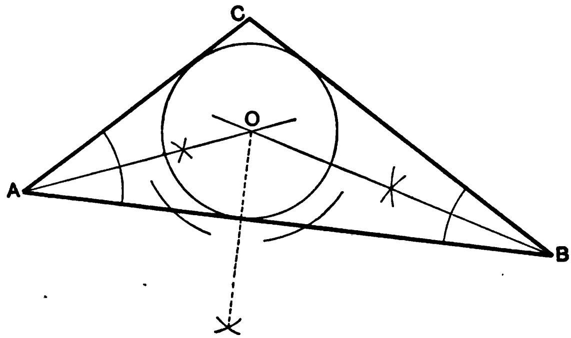
Q. E. D.

213. *Remark.*—The foregoing method is especially convenient when the given point B is near the edge of the paper.

214. *Def.*—A circle is said to be **inscribed** in a polygon, if it be tangent to every side of the polygon. In the same case, the polygon is said to be **circumscribed** about the circle.



215. CONSTRUCTION. *To inscribe a circle in a given triangle.*



GIVEN	the triangle ABC.
TO CONSTRUCT	an inscribed circle.

Bisect two of the angles, as A and B.

With O, the intersection of these bisectors, as a centre and the distance to any side as a radius, describe a circumference. This gives the circle required.

Proof.—O lies in AO, and is therefore equally distant from AC and AB.

O lies in BO, and is therefore equally distant from BC and BA. § 108

[The bisector of an angle is the locus of points equally distant from its sides.]

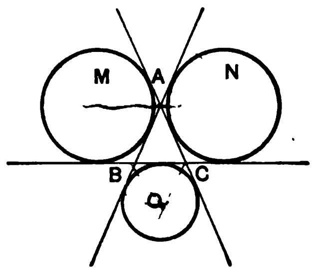
Therefore O is equally distant from *all* sides.

Hence the circle described with O as a centre, and with this distance as a radius, will be tangent to the three sides. § 173

Q. E. D.

216. *Def.*—**Escribed circles** are circles which are tangent to one side of a triangle and the other two sides produced.

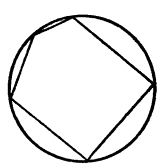
Thus, for the triangle *ABC*, *M*, *N*, and *O* are escribed circles.



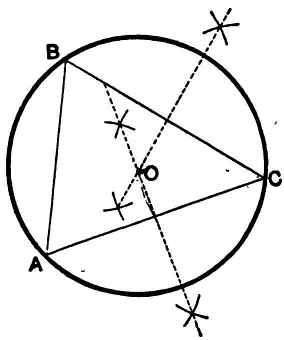
217. *Exercise.*—Construct the three escribed circles of a given triangle.

Hint.—Find centres, as in § 215.

218. *Def.*—A circle is said to be **circumscribed** about a polygon, if the circumference of the circle passes through every vertex of the polygon. In the same case, the polygon is said to be **inscribed** in the circle.



219. CONSTRUCTION. *To circumscribe a circle about a given triangle.*



GIVEN the triangle ABC.

TO CONSTRUCT a circumscribed circle.

Draw the perpendicular bisectors of two of the sides BC and AC.

With O their intersection as a centre, and the distance to any vertex as a radius, describe a circumference.

This gives the circle required.

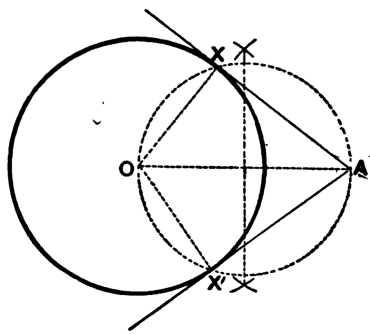
Proof.—O is equally distant from B and C.
O is equally distant from A and C. } § 103

[The perpendicular bisector is the locus of points equally distant from the extremities of a straight line.]

Therefore O is equally distant from *all* vertices, and the circle described as above is the required circle. Q. E. D.

220. *Remark.*—The foregoing construction also enables us to draw a circumference through three points *not in the same straight line* or to find the centre of a given circumference or arc. § 166

221. CONSTRUCTION. *To construct a tangent to a given circle from a given point without.*



GIVEN the circle O and the point A without.

TO CONSTRUCT from A a tangent to the circle.

Upon AO as a diameter construct a circumference intersecting the given circumference at X and X'.

Join AX and AX'.

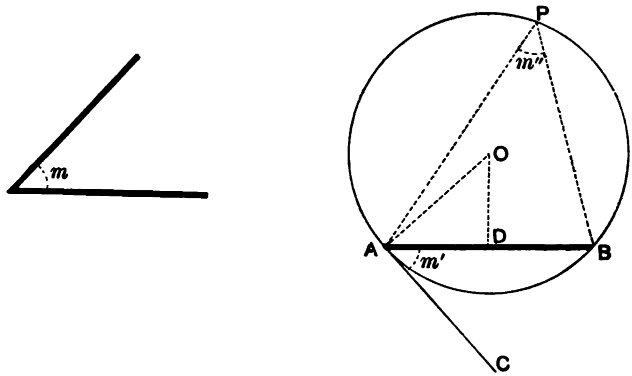
These lines are the required tangents.

Proof.—Angle AXO is a right angle. § 202
[Being inscribed in a semicircle.]

Hence AX is a tangent to the circle O. § 173
[Being perpendicular to a radius at its extremity.]

Likewise AX' is tangent. Q. E. D.

222. CONSTRUCTION. *Upon a given straight line to construct a segment which shall contain a given angle.*



GIVEN the straight line AB and the angle m.

TO CONSTRUCT—a segment upon AB which shall contain an angle equal to m.

At A construct m' equal to m, and having AB as one of its sides. § 80

Draw AO perpendicular to AC, and DO perpendicularly bisecting AB.

With O, the intersection of these two lines, as a centre, and OA or OB as a radius, construct a segment APB. This is the segment required.

Proof.—CA is tangent to the circle. § 173

[Being perpendicular to a radius at its extremity.]

Therefore m' is measured by $\frac{1}{2}$ arc AB § 205

But m'' (any angle inscribed in the segment) is also measured by $\frac{1}{2}$ arc AB. § 197

Therefore $m' = m''$. Ax. 1

But $m = m'$. Cons.

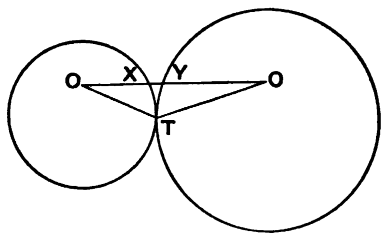
Therefore $m = m''$. Ax. 1

Q. E. D.

PROBLEMS OF DEMONSTRATION

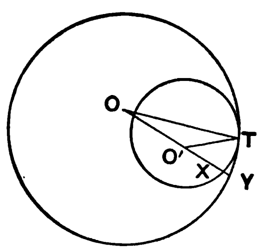
223. *Defs.*—Two circles are **tangent** which touch at but one point. They may be tangent **internally,** so that one circle is within the other; or **externally,** so that each is without the other.

224. *Exercise.*—The straight line joining the centres of two circles tangent externally passes through the point of tangency.



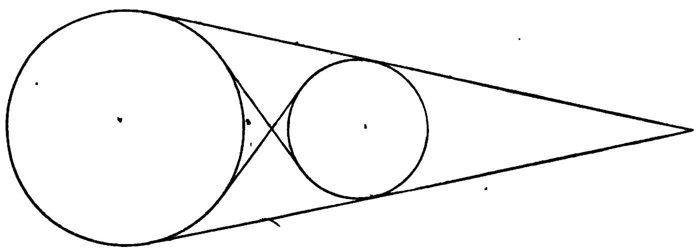
Hint.—Suppose OO' not through T, and prove OO' greater than and also less than the sum of the radii.

225. *Exercise.*—The straight line joining the centres of two circles internally tangent passes through the point of tangency.



Hint.—If not, prove the distance between centres greater than and also less than the difference of the radii.

226. *Defs.*—If each of two circles is entirely without the other, four common tangents can be drawn. Two of these are called external, and two internal. An **external tangent** is one such that the two circles lie on the same side of it; an **internal tangent** is one such that the two circles lie on opposite sides of it.



Question.—In case the two circles are themselves tangent externally, how many common tangents of each kind can be drawn? In case the two circles overlap? In case they are tangent internally? In case one is within the other?

227. *Exercise.*—The two common external tangents to two circles meet the line joining their centres in the same point. Also the two common internal tangents meet the line of centres in the same point.

228. *Exercise.*—The sum of two opposite sides of a quadrilateral circumscribed about a circle is equal to the sum of the other two sides (§ 176).

229. *Exercise.*—The sum of two opposite angles of a quadrilateral inscribed in a circle is equal to the sum of the other two angles, and is equal to two right angles.

230. *Exercise.*—Two circles are tangent externally at A. The line of centres contains A, by § 224. Prove (1) that the perpendicular to the line of centres at A is a common tangent; (2) that it bisects the other two common tangents; and (3) that it is the locus of all points from which tangents drawn to the two circles are equal.

231. *Exercise.*—Find the locus of the middle points of all chords of a given length.

232. *Exercise.*—If a straight line be drawn through the point of contact of two tangent circles forming chords, the radii drawn to the remaining extremities of these chords are parallel. Also, the tangents at these extremities are parallel. What two cases are possible?

PROBLEMS OF CONSTRUCTION

233. *Exercise.*—Draw a straight line tangent to a given circle and parallel to a given straight line.

234. *Exercise.*—Construct a right triangle, given the hypotenuse and an acute angle.

235. *Exercise.*—Construct a right triangle, given the hypotenuse and a side.

236. *Exercise.*—Construct a right triangle, given the hypotenuse and the distance of the hypotenuse from the vertex of the right angle.

237. *Exercise.*—Construct a circle tangent to a given straight line and having its centre in a given point.

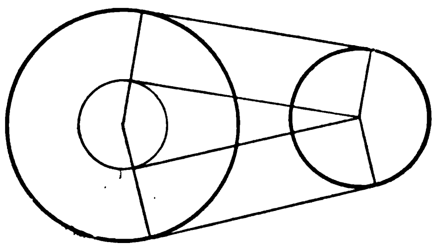
238. *Exercise.*—Construct a circumference having its centre in a given line and passing through two given points.

239. *Exercise.*—Find the locus of the centres of all circles of given radius tangent to a given straight line.

240. *Exercise.*—Construct a circle of given radius tangent to two given straight lines.

241. *Exercise.*—Construct a circle of given radius tangent to two given circles.

242. *Exercise.*—Construct all the common tangents to two given circles.



Hint.—For the external tangents draw a circle with radius equal to the difference of the radii of the given circles and its centre at the centre of the larger circle. Draw tangents to this circle from the centre of the smaller circle.

PLANE GEOMETRY

BOOK III

PROPORTION AND SIMILAR FIGURES

243. *Def.*—A **proportion** is an equality of ratios.

Thus, if the ratio $\frac{A}{B}$ is equal to the ratio $\frac{C}{D}$, then the equality $\frac{A}{B}=\frac{C}{D}$ constitutes a proportion.

This may also be written

$$A:B=C:D, \text{ or } A:B::C:D,$$

and is read, A is to B as C is to D.

244. *Def.*—The four magnitudes A, B, C, D are called the **terms** of the proportion.

245. *Defs.*—The first and last terms are the **extremes,** the second and third, the **means.**

246. *Defs.*—The first and third terms are called the **antecedents,** and the second and fourth, the **consequents.**

247. THEOREM. *If four quantities are in proportion, their numerical measures are in proportion; and conversely.*

GIVEN $$\frac{A}{B}=\frac{C}{D}.$$

TO PROVE: $\frac{a}{b}=\frac{c}{d}$, where a, b, c, d are the numerical measures of A, B, C, D, respectively.

Now $$\frac{A}{B}=\frac{a}{b} \text{ and } \frac{C}{D}=\frac{c}{d}. \qquad \S\ 180$$

[The ratio of two quantities is equal to the ratio of their numerical measures.]

Whence $$\frac{a}{b}=\frac{c}{d}. \qquad \text{Ax. } 1$$

Q. E. D.

CONVERSELY: If $\frac{a}{b}=\frac{c}{d}$, then $\frac{A}{B}=\frac{C}{D}$. This can be proved in like manner.

248. *Remark.*—In order that the preceding theorems shall hold true, A and B must be quantities of the *same kind*, as two straight lines, or two angles, and C and D also of the same kind; *but it is not necessary that A and B shall be of the same kind as C and D.*

249. *Def.*—One variable quantity is said to be **proportional** to another, when any two values of the first have the same ratio as two corresponding values of the second.

Thus, Proposition XI., Book II., may be expressed:

An angle at the centre of a circle is proportional to its intercepted arc.

By this we mean that the ratio of a given angle, as AOB, to some other angle, as $A'O'B'$, is equal to the ratio of the corresponding arcs, AB and $A'B'$.

TRANSFORMATION OF PROPORTIONS

250. THEOREM. *If four numbers are in proportion, the product of the extremes equals the product of the means.*

GIVEN $$\frac{a}{b}=\frac{c}{d}. \quad (1)$$

TO PROVE $$ad=bc. \quad (2)$$

Clear (1) of fractions, i. e., multiply both sides by bd, the product of the denominators of (1).

We have $$ad=bc. \quad (2) \qquad \text{Ax. } 7$$

Q. E. D.

251. THEOREM. Conversely, *if the product of two numbers equals the product of two others, either pair may be made the extremes and the other pair the means of a proportion.*

GIVEN $ad = bc.$ (2)

TO PROVE $\frac{a}{b} = \frac{c}{d}.$ (1)

Divide both sides of (2) by bd, the product of the denominators of (1).

We have $\frac{a}{b} = \frac{c}{d}.$ (1) Ax. 8

Q. E. D.

Again,

GIVEN $bc = ad.$ (2)

TO PROVE $\frac{b}{a} = \frac{d}{c}.$ (3)

Dividing (2) by ac, the product of the denominators of (3), we obtain (3). Q. E. D.

Question.—By dividing the equation $ad = bc$ by the product of two of the letters, one being from each side, how many proportions in all can be obtained? Write them. If the equation be written $bc = ad$, how many can be obtained, and how do they differ from the former set?

252. *Remark.*—The student has already noticed that the process by which equation (1) was obtained from (2) was the reverse of that by which (2) was obtained from (1). Also it is easy to see that (3) was obtained from (2) by a process the reverse of that by which (2) could have been obtained from (3). Now it is always much easier to see how an equation can be reduced to $ad = bc$ than to see how it can be deduced from $ad = bc$. Since the latter is the reverse of the former, we have the following practical guide for obtaining a required equation from $ad = bc$: First see what processes would be necessary if you wished to reduce the equation to $ad = bc$; reverse these steps in order, and you have the method required.

The preceding rule will be better understood from the following example:

253. If $ad=bc$ (2), prove $\frac{a+b}{b}=\frac{c+d}{d}$. (5)

As it is not at first evident what operations to perform on (2) to obtain (5), let us see what would be necessary in the reverse proof. These operations, as the student will easily see, would be:

Step 1.—Clear (5) of fractions, i. e., multiply both sides by bd.

Step 2.—Cancel bd, i. e., subtract bd from both sides.

By the rule of § 252 we need to reverse these steps, viz.:

First, add bd to both sides of (2).

This gives $$ad+bd=bc+bd. \qquad \text{Ax. 2}$$

Secondly, divide both sides by bd.

This gives $$\frac{a+b}{b}=\frac{c+d}{d}. \quad (5) \qquad \text{Ax. 8}$$

254. THEOREM. *If four numbers are in proportion, they are also in proportion by inversion.*

GIVEN $$\frac{a}{b}=\frac{c}{d}. \quad (1)$$

TO PROVE $$\frac{b}{a}=\frac{d}{c}. \quad (3)$$

OUTLINE PROOF.—Derive from (1) equation (2), or $bc=ad$, and from (2) equation (3) by the rule of § 252.

255. *Exercise.*—Prove § 254 otherwise.

256. THEOREM. *If four numbers are in proportion, they are also in proportion by alternation.*

GIVEN $$\frac{a}{b}=\frac{c}{d}. \quad (1)$$

TO PROVE $$\frac{a}{c}=\frac{b}{d}. \quad (4)$$

Hint.—Proceed as in § 254, or multiply each side of (1) by $\frac{b}{c}$.

257. THEOREM. *If four numbers are in proportion, they are also in proportion by composition.*

GIVEN $$\frac{a}{b}=\frac{c}{d}. \qquad (1)$$

TO PROVE $$\frac{a+b}{b}=\frac{c+d}{d}. \qquad (5)$$

Hint.—Proceed as in § 254, or add 1 to each side of equation (1).

258. *Exercise.*—If $\frac{a}{b}=\frac{c}{d}$, prove $\frac{a+b}{a}=\frac{c+d}{c}$.

259. THEOREM. *If four numbers are in proportion, they are also in proportion by division.*

GIVEN $$\frac{a}{b}=\frac{c}{d}. \qquad (1)$$

TO PROVE $$\frac{a-b}{b}=\frac{c-d}{d}. \qquad (6)$$

Hint.—Proceed as in § 254, or subtract 1 from each side of equation (1).

260. *Exercise.*—If $\frac{a}{b}=\frac{c}{d}$, prove $\frac{a-b}{a}=\frac{c-d}{c}$.

261. THEOREM. *If four numbers are in proportion, they are also in proportion by composition and division.*

GIVEN $$\frac{a}{b}=\frac{c}{d}. \qquad (1)$$

TO PROVE $$\frac{a+b}{a-b}=\frac{c+d}{c-d}. \qquad (7)$$

Hint—Divide equation (5) by (6), or proceed as in § 254.

262. THEOREM. *If four numbers are in proportion, equimultiples of the antecedents will be in proportion with equimultiples of the consequents.*

GIVEN $$\frac{a}{b}=\frac{c}{d}. \quad (1)$$

TO PROVE $$\frac{ma}{nb}=\frac{mc}{nd}. \quad (8)$$

Hint.—This is proved by multiplying each side of (1) by $\frac{m}{n}$.

263. *Remark.*—The equations so far considered are

$$\frac{a}{b}=\frac{c}{d} \quad (1)$$

$$ad=bc \quad (2)$$

$$\frac{b}{a}=\frac{d}{c} \quad (3)$$

$$\frac{a}{c}=\frac{b}{d} \quad (4)$$

$$\frac{a+b}{b}=\frac{c+d}{d} \quad (5)$$

$$\frac{a-b}{b}=\frac{c-d}{d} \quad (6)$$

$$\frac{a+b}{a-b}=\frac{c+d}{c-d} \quad (7)$$

$$\frac{ma}{nb}=\frac{mc}{nd}. \quad (8)$$

The student will see that, if any one of these equations be given, all the others can be obtained. For the given equation can be transformed into (2), and (2) into any other by the method of § 252.

264. *Def.*—A **continued proportion** is an equality of three or more ratios; as

$$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{h}{k}=\text{etc.}$$

265. THEOREM. *In a continued proportion the sum of any number of antecedents is to the sum of the corresponding consequents as any antecedent is to its consequent.*

GIVEN $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{h}{k}=\text{etc.}$

TO PROVE $\frac{a+c+e}{b+d+f}=\frac{a}{b}=\frac{c}{d}=\text{etc.}$

Call each one of the equal ratios $\frac{a}{b}$, $\frac{c}{d}$, etc., r.

Then $\frac{a}{b}=r$, or $a=br$. Ax. 7

$\frac{c}{d}=r$, or $c=dr$.

$\frac{e}{f}=r$, or $e=fr$.

Adding these equations together, we have

$$a+c+e=br+dr+fr=r\,(b+d+f). \qquad \text{Ax. 2}$$

Dividing both sides by $b+d+f$ gives

$$\frac{a+c+e}{b+d+f}=r. \qquad \text{Ax. 8}$$

But $r=\frac{a}{b}=\frac{c}{d}=\text{etc.}$

Therefore $\frac{a+c+e}{b+d+f}=\frac{a}{b}=\frac{c}{d}=\text{etc.}$ Ax. 1

Q. E. D.

266. THEOREM. *The products of the corresponding terms of any number of proportions from a proportion.*

GIVEN
$$\begin{cases} \dfrac{a}{b}=\dfrac{c}{d}, \\ \dfrac{a'}{b'}=\dfrac{c'}{d'}, \\ \dfrac{a''}{b''}=\dfrac{c''}{d''}, \\ \text{etc.} \end{cases}$$

TO PROVE
$$\frac{aa'a''}{bb'b''}=\frac{cc'c''}{dd'd''}.$$

Multiply all the given equations together.

The result is
$$\frac{aa'a''}{bb'b''}=\frac{cc'c''}{dd'd''}.$$
Q. E. D.

267. THEOREM. *If four numbers are in proportion, like powers of these numbers are in proportion.*

GIVEN
$$\frac{a}{b}=\frac{c}{d}.$$

TO PROVE
$$\frac{a^2}{b^2}=\frac{c^2}{d^2};\quad \frac{a^3}{b^3}=\frac{c^3}{d^3};\quad \frac{a^4}{b^4}=\frac{c^4}{d^4};\quad \text{etc.}$$

This is proved by raising the two sides of the given equation to the required power.

268. *Def.*—The **segments** of a straight line are the parts into which it is divided.

269. *Def.*—Two straight lines are **divided proportionally,** when the ratio of one line to either of its segments is equal to the ratio of the other line to its corresponding segment.

PROPOSITION I. THEOREM

270. *A straight line parallel to one side of a triangle divides the other two sides proportionally.*

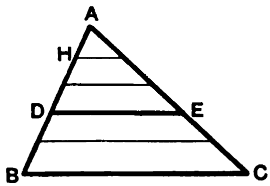


FIG. 1

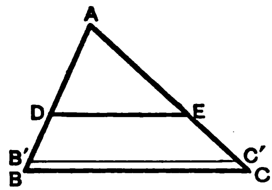


FIG. 2

GIVEN—the straight line DE parallel to the side BC of the triangle ABC.

TO PROVE $$\frac{AB}{AD}=\frac{AC}{AE}.$$

CASE I.—*When AB and AD are commensurable* (Fig. 1).

Let AH be the unit of measure, and suppose it is contained in AB five times, and in AD three times.

Then $$\frac{AB}{AD}=\frac{5}{3}. \quad (1)$$ § 180

Through the several points of division on AB and AD draw lines parallel to BC.

These lines will divide AC into five equal parts, of which AE contains three. § 127

[If any number of parallels intercept equal parts on one cutting line, they will intercept equal parts on every other cutting line.]

Therefore $$\frac{AC}{AE}=\frac{5}{3}. \quad (2)$$ § 180

Comparing (1) and (2),

$$\frac{AB}{AD}=\frac{AC}{AE}.$$ Ax. 1

Q. E. D.

CASE II. *When AB and AD are incommensurable* (Fig. 2).

Let AD be divided into any number of equal parts, and let one of these parts be applied to AB as a measure.

Since AD and AB are incommensurable, a certain number of these parts will extend from A to B', leaving a remainder BB' less than one of these parts.

Through B' draw $B'C'$ parallel to BC.

Since AD and AB' are commensurable,

$$\frac{AB'}{AD}=\frac{AC'}{AE}. \qquad \text{Case I}$$

Now, suppose the number of divisions of AD to be indefinitely increased.

Then each division, either of AD or of AE, can be made as small as we please.

Hence $B'B$ and $C'C$, being always less than one of these divisions, can be made as small as we please.

Hence $\left.\begin{array}{l} AB' \text{ approaches } AB \text{ as a limit.} \\ AC' \text{ approaches } AC \text{ as a limit.}\end{array}\right\}$ § 185

Hence $\left.\begin{array}{l} \dfrac{AB'}{AD} \text{ approaches } \dfrac{AB}{AD} \text{ as a limit.} \\ \dfrac{AC'}{AE} \text{ approaches } \dfrac{AC}{AE} \text{ as a limit.}\end{array}\right\}$ § 190

But we proved $\dfrac{AB'}{AD}=\dfrac{AC'}{AE}.$

Hence $\dfrac{AB}{AD}=\dfrac{AC}{AE}.$ § 186

Q. E. D.

271. COR. I. $\dfrac{AD}{DB}=\dfrac{AE}{EC}.$

Hint—This is proved by division and inversion.

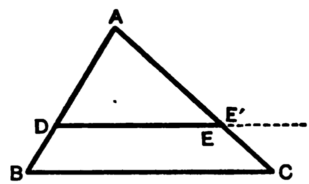
272. COR. II. $$\frac{AB}{AC}=\frac{AD}{AE}=\frac{DB}{EC}.$$

Hint.—This is proved by alternation.

PROPOSITION II. THEOREM

273. *If a straight line divides two sides of a triangle proportionally, it is parallel to the third side.*

[Converse of Proposition I.]



GIVEN—the straight line DE, in the triangle ABC, so drawn that

$$\frac{AB}{AD}=\frac{AC}{AE}.$$

TO PROVE DE parallel to BC.

From D draw DE' parallel to BC.

Then $$\frac{AB}{AD}=\frac{AC}{AE'}.$$ § 270

[A straight line parallel to one side of a triangle divides the other two sides proportionally.]

But $$\frac{AB}{AD}=\frac{AC}{AE}.$$ Hyp.

Hence $$\frac{AC}{AE}=\frac{AC}{AE'}.$$ Ax. 1

The numerators of these equal fractions being equal, their denominators must also be equal. § 254, Ax. 7

That is, $AE=AE'$.

Therefore E and E' coincide.

Hence DE and DE' coincide. Ax. a

But DE' is parallel to BC by construction.

Therefore DE, which coincides with DE', is parallel to BC.

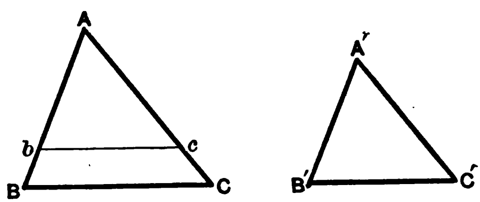
Q. E. D.

274. *Def.*—**Similar polygons** are polygons which have the angles of one equal to the angles of the other, each to each, and the corresponding, or **homologous**, sides proportional.*

As we shall see, if the polygons are triangles, neither of these two conditions can be true without the other; but, if the polygons have four or more sides, either can be true without the other.

PROPOSITION III. THEOREM

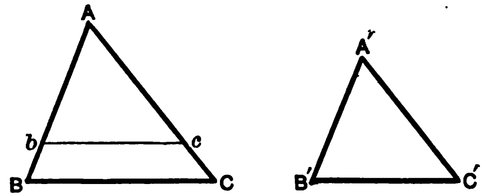
275. *Two triangles which are mutually equiangular are similar.*



GIVEN—in the triangles ABC and $A'B'C'$, the angles A, B, and C, equal respectively to the angles A', B', C'.

TO PROVE the triangle ABC similar to $A'B'C'$.

* There is some evidence that the early Egyptians knew of the properties of similar figures. But the first philosopher who is mentioned as employing them is Thales (600 B.C.). One of his simplest calculations was to find the height of a building by measuring its shadow at that hour of the day when a man's shadow is of the same length as himself.



Apply the triangle $A'B'C'$ to ABC so that the angle A' shall fall on A.

Then the triangle $A'B'C'$ will take the position Abc.

Since the angle Abc (or the angle B') is given equal to B, bc is parallel to BC. § 44

[If two straight lines are cut by a third, so that corresponding angles are equal, the straight lines are parallel.]

Hence $$\frac{AB}{Ab}=\frac{AC}{Ac}.$$ § 270

or $$\frac{AB}{A'B'}=\frac{AC}{A'C'}.$$

By applying the triangle $A'B'C'$ to ABC so that B' shall coincide with its equal B, it may be shown in the same manner that $$\frac{AB}{A'B'}=\frac{BC}{B'C'}.$$

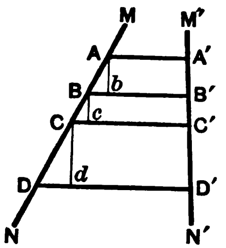
Therefore $$\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}.$$ Ax. 1

Hence the homologous sides are proportional and the triangles are similar. § 274

Q. E. D.

276. COR. I. *If two triangles have two angles of one equal to two angles of the other, they are similar.*

277. COR. II. *If two straight lines are cut by a series of parallels, the corresponding segments of the two lines are proportional.*

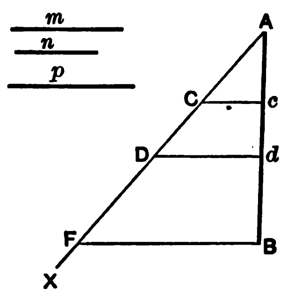


Hint.—Let MN and $M'N'$ be cut by the parallels AA', BB', CC', and DD'.

Draw Ab, Bc, and Cd parallel to $M'N'$.

Prove the triangles ABb, BCc, and CDd similar.

278. CONSTRUCTION. *To divide a given straight line into parts proportional to given straight lines.*



Required.—To divide AB into parts proportional to m, n, and p.

From A draw an indefinite straight line AX, upon which lay off $AC=m$, $CD=n$, and $DF=p$.

Join FB and draw Dd and Cc parallel to FB.

Ac, cd, and dB will then be proportional to m, n, and p. § 277

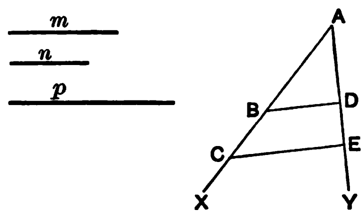
Q. E. F.

279. *Remark.*—If the lines m, n, and p are equal to each other, the line AB will be divided into equal parts. (See also § 127.)

280. *Def.*—A **fourth proportional** to three given quantities is the fourth term of a proportion whose first three terms are the three given quantities taken in order.

281. *Defs.*—When the two means of a proportion are equal, either of them is said to be a **mean proportional** between the other two terms. The fourth term in this case is called a **third proportional** to the other two.

282. CONSTRUCTION. *To find a fourth proportional to three given straight lines.*



Required.—To find a fourth proportional to m, n, and p.

Draw from A the two indefinite lines AX and AY.

Lay off $AB=m$, $AD=n$, and $AC=p$.

Join BD, and through C draw CE parallel to BD.

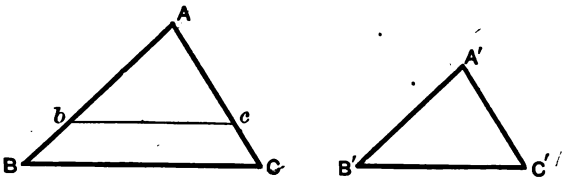
Then AE will be the fourth proportional.

For $$\frac{AB}{AD}=\frac{AC}{(AE)}.$$ § 272

283. *Remark.*—If n and p are equal, then also AC and AD are equal, and AE is a third proportional to AB and AD.

PROPOSITION IV. THEOREM

284. *Two triangles are similar when their homologous sides are proportional.*



GIVEN—in the two triangles ABC and $A'B'C'$,

$$\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}.$$

TO PROVE the triangle ABC similar to $A'B'C'$.

On AB lay off $Ab=A'B'$, and on AC lay off $Ac=A'C'$, and join bc.

Then by substituting Ab and Ac for their equals $A'B'$ and $A'C'$ in the given proportion, we have

$$\frac{AB}{Ab}=\frac{AC}{Ac}.$$

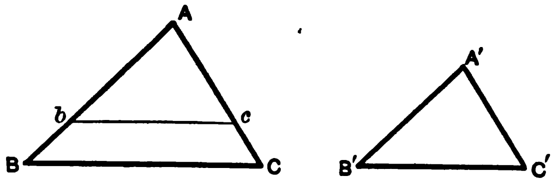
Therefore the line bc is parallel to BC. § 273

[If a straight line divides two sides of a triangle proportionally, it is parallel to the third side.]

And the angle Abc = the angle B, and $Acb=C$. § 49

Hence the triangles ABC and Abc, being mutually equiangular, are similar. § 275

It remains to show that the triangle Abc equals the triangle $A'B'C'$. Since two of their sides are given equal, we only need to show that the third sides bc and $B'C'$ are equal.



Now $\dfrac{bc}{BC}=\dfrac{Ab}{AB}=\dfrac{A'B'}{AB}$. § 274

But $\dfrac{B'C'}{BC}=\dfrac{A'B'}{AB}$. Hyp.

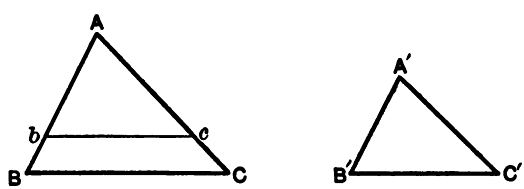
Hence $\dfrac{bc}{BC}=\dfrac{B'C'}{BC}$. Ax. 1

Hence $bc=B'C'$. § 254, Ax. 7

Therefore the triangles Abc and $A'B'C'$ are equal. § 89
But the triangle Abc has been proved similar to ABC.
Hence $A'B'C'$, the equal of Abc, is similar to ABC. Q. E. D.

PROPOSITION V. THEOREM

285. *Two triangles are similar when an angle of the one is equal to an angle of the other, and the sides including these angles are proportional.*



GIVEN—in the triangles ABC and $A'B'C'$, the angle $A=A'$ and

$$\frac{AB}{A'B'}=\frac{AC}{A'C'}.$$

TO PROVE the triangles similar.

Place the triangle $A'B'C'$ on ABC so that the angle A' shall coincide with A, and B' fall at b, and C' at c.

Then $$\frac{AB}{Ab}=\frac{AC}{Ac}.$$ Hyp.

Therefore bc is parallel to BC, § 273

[If a straight line divides two sides of a triangle proportionally, it is parallel to the third side.]

and the angles b and c are equal respectively to B and C. § 49

Hence the triangles ABC and Abc are similar. § 275

[Two triangles which are mutually equiangular are similar.]

But Abc is equal to $A'B'C'$.

Therefore the triangle $A'B'C'$ is also similar to ABC. Q. E. D.

PROPOSITION VI. THEOREM

286. *Two triangles which have their sides parallel each to each, or perpendicular each to each, are similar.*

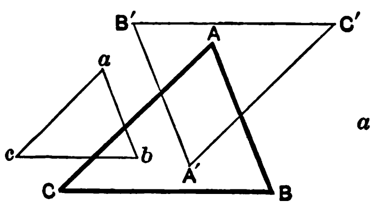


FIG. 1

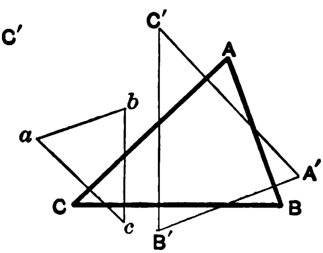


FIG. 2

GIVEN—in the triangles $A'B'C'$ and ABC, that the sides $A'B'$, $A'C'$, and $B'C'$, are respectively parallel to AB, AC, and BC in Fig. 1, and perpendicular in Fig. 2.

TO PROVE the triangles similar.

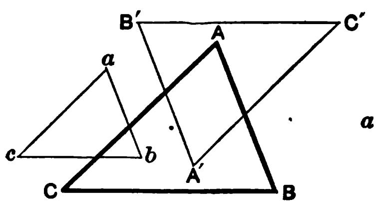


FIG. 1

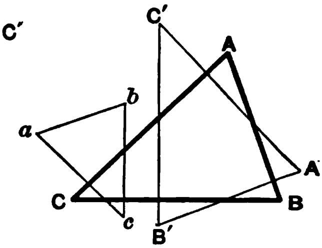


FIG. 2

Since the sides of the two triangles in Fig. 1 are parallel, and in Fig. 2 are perpendicular each to each, the included angles formed by each pair of sides are in both cases either equal or supplementary. §§ 51, 53

Hence, in both cases, we can make three hypotheses, as follows:

1st hypothesis, $A+A'=2$ right angles; $B+B'=2$ right angles; $C+C'=2$ right angles.

2d hypothesis, $A=A'$; $B+B'=2$ right angles; $C+C'=2$ right angles.

3d hypothesis, $A=A'$; $B=B'$; and hence also $C=C'$. § 61

Neither the first nor the second of these hypotheses can be true, for then the sum of the angles of a triangle would be more than two right angles. § 58

Therefore the third is the only one admissible.

Hence the two triangles are similar. Q. E. D.

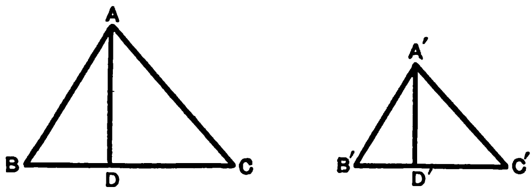
287. *Remark.*—The student will observe that ABC and abc can be proved similar in the same manner.

288. *Remark.*—The homologous sides in the two triangles are any two parallel sides (Fig. 1) or any two perpendicular sides (Fig. 2).

289. *Defs.*—The **base** of a triangle is that side upon which the triangle is supposed to stand. The **altitude** is the perpendicular to the base from the opposite vertex.

PROPOSITION VII. THEOREM

290. *In two similar triangles, corresponding altitudes have the same ratio as any two homologous sides.*



GIVEN—two similar triangles ABC and $A'B'C'$, AD and $A'D'$ being their corresponding altitudes.

TO PROVE $$\frac{AD}{AD'}=\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}.$$

The two right triangles ABD and $A'B'D'$ are similar, since B and B' are equal angles, and ADB and $A'D'B'$ are both right angles. § 276

[If two triangles have two angles of one equal to two angles of the other, they are similar.]

Then $$\frac{AD}{A'D'}=\frac{AB}{A'B'}.$$ § 274

But, since the triangles ABC and $A'B'C'$ are similar, we have $$\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}.$$ § 274

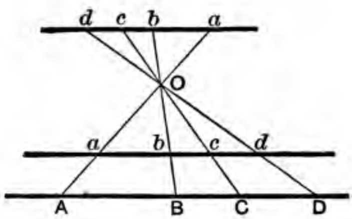
Hence $$\frac{AD}{A'D'}=\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}.$$ Ax. 1

Q. E. D.

5

PROPOSITION VIII. THEOREM

291. *If three or more straight lines drawn through a common point intersect two parallels, the corresponding segments of the parallels are proportional.*



GIVEN—the lines OA, OB, OC, OD, drawn through a common point O and intersecting the parallels AD and ad in the points A, B, C, D, and a, b, c, d.

TO PROVE $$\frac{AB}{ab}=\frac{BC}{bc}=\frac{CD}{cd}.$$

Since ad is parallel to AD,
angle Oab = angle OAB, and angle Oba = angle OBA. §§ 48, 49

Therefore the triangle aOb is similar to AOB. § 276

[If two triangles have two angles of one equal to two angles of the other, they are similar.]

In the same way the triangles bOc and cOd are similar respectively to BOC and COD.

Therefore $$\frac{ab}{AB}=\left(\frac{Ob}{OB}\right)=\frac{bc}{BC}=\left(\frac{Oc}{OC}\right)=\frac{cd}{CD}.$$ § 274

Whence $$\frac{ab}{AB}=\frac{bc}{BC}=\frac{cd}{CD}.$$ Ax. 1

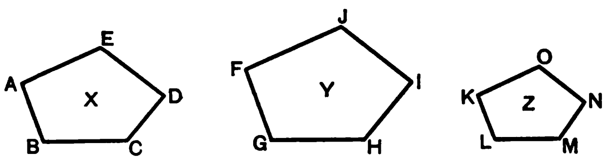
Q. E. D.

292. COR. If $AB = BC = CD$, then $ab = bc = cd$. Therefore *the lines, drawn from the vertex of a triangle dividing the base into equal parts, divide a parallel to the base into equal parts also.*

293. *Exercise.*—Two men, on opposite sides of a street, walk in opposite directions, and so that a tree between them always hides each from the other. Prove that, if one man walks uniformly, the other must also, and show the connection between the position of the tree and the ratio of their speeds.

PROPOSITION IX. THEOREM

294. *Two polygons similar to a third are similar to each other.*



GIVEN the polygons X and Y, both similar to Z.

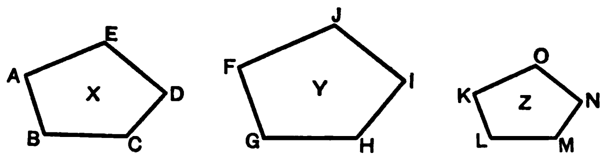
TO PROVE that X and Y are similar to each other.

Angles A and F are each equal to K. Hyp.

Therefore they are equal to each other. Ax. 1

In like manner the angles B, C, D, E of X are equal to the corresponding angles of G, H, I, J of Y.

$$\left.\begin{aligned} &\text{Again} && \frac{AB}{KL}=\frac{BC}{LM}=\frac{CD}{MN}=\text{etc.,} \\ &\text{and} && \frac{FG}{KL}=\frac{GH}{LM}=\frac{HI}{MN}=\text{etc.} \end{aligned}\right\} \qquad \S\ 274$$



Dividing the first set of equations by the second,

$$\frac{AB}{FG}=\frac{BC}{GH}=\frac{CD}{HI}=\text{etc.}$$

Therefore X and Y are similar. § 274

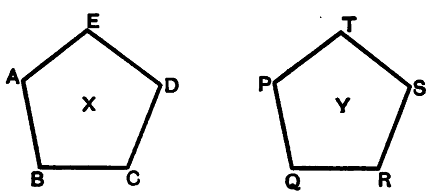
[Having their angles respectively equal and their homologous sides proportional.] Q. E. D.

295. *Def.*—The **ratio of similitude** of any two similar polygons is the ratio of any two homologous sides.

[Thus in § 294 the ratio of AB to FG is the ratio of similitude of X and Y.]

PROPOSITION X. THEOREM

296. *Two similar polygons are equal if their ratio of similitude is unity.*



GIVEN—the similar polygons X and Y, whose ratio of similitude is unity.

TO PROVE X and Y equal.

The angles of X and Y are respectively equal. § 274

Again $$\frac{AB}{PQ}=1.$$ Hyp.

Therefore $AB=PQ$; likewise $BC=QR$; etc.

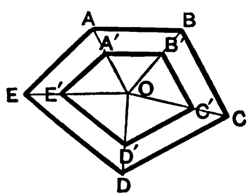
That is, the sides of X and Y are respectively equal.

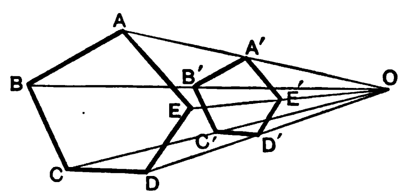
Hence the polygons, having their corresponding angles and sides respectively equal, can be made to coincide and are equal. Q. E. D.

297. *Defs.*—If the vertices A, B, C, D, etc., of a polygon are joined by straight lines to a point O, and the lines OA, OB, OC, OD, etc., are divided in a given ratio at the points A', B', C', D', etc., the polygon $A'B'C'D'$ etc., is said to be **radially situated** with respect to the polygon $ABCD$, etc.

The ratio of the lines OA' and OA is called the **determining ratio** of the two polygons.

The point O is called the **ray centre.**





In each of the figures the vertices A and A', B and B', C and C', etc., lie on the rays OA, OB, OC, etc., making

$$\frac{OA}{OA'}=\frac{OB}{OB'}=\frac{OC}{OC'}=\text{etc.}$$

The two polygons, $ABCDE$ and $A'B'C'D'E'$, are therefore radially situated.

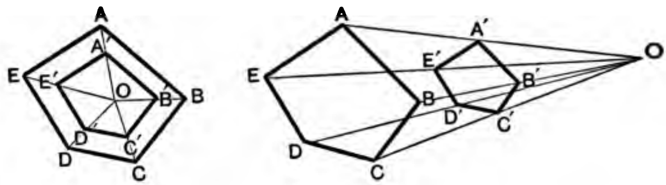
The points A', B', C', D' are **homologous** to the points A, B, C, D respectively.

Straight lines determined by homologous points are **homologous.**

Angles formed by homologous lines are **homologous.**

PROPOSITION XI. THEOREM

298. *Two polygons radially situated are similar and their ratio of similitude is equal to the determining ratio.*



GIVEN—the polygons $ABCDE$ and $A'B'C'D'E'$ radially situated, O being the ray centre.

TO PROVE—they are similar, and that the determining ratio is their ratio of similitude.

AB is parallel to $A'B'$, BC to $B'C'$, etc. § 273

[If a straight line divide two sides of a triangle proportionally, it is parallel to the third side.]

Hence angle $ABC = A'B'C'$, angle $BCD = B'C'D'$, etc. § 51

[Having their sides respectively parallel and in the same right-and-left order.]

Again, triangle OAB is similar to $OA'B'$, OBC to $OB'C'$, etc. § 285

Therefore $$\frac{AB}{A'B'} = \left(\frac{OB}{OB'}\right) = \frac{BC}{B'C'} = \left(\frac{OC}{OC'}\right) = \text{etc.} \qquad \S\ 274$$

Whence $$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \text{etc.} \qquad \text{Ax. 1}$$

Since the polygons have their angles respectively equal and their homologous sides proportional, they are similar. § 274

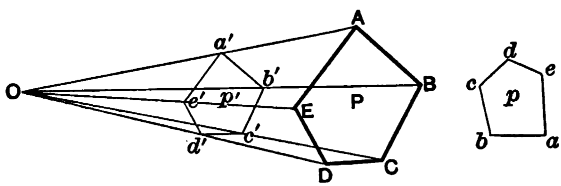
Also, their ratio of similitude $\frac{AB}{A'B'}$ = determining ratio $\frac{OB}{OB'}$.

Q. E. D.

299. *Def.*—The ray centre is also called the **centre of similitude.**

PROPOSITION XII. THEOREM

300. *Any two similar polygons can be radially placed, the determining ratio being equal to the ratio of similitude.*



GIVEN the similar polygons P and p.

TO PROVE—that they can be radially placed, the determining ratio being the ratio of similitude.

With any point O as ray centre form a polygon p' radially situated with regard to P, having the determining ratio $\frac{Oa'}{OA}$ equal to the ratio of similitude $\frac{ab}{AB}$ of p and P.

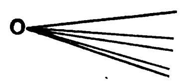
Then p' and P will be similar, the ratio of similitude being

$$\frac{a'b'}{AB} = \frac{Oa'}{OA}. \qquad \S\ 298$$

But p and P are given similar, and their ratio of similitude is

$$\frac{ab}{AB}.$$

Therefore p' and p are similar. § 294

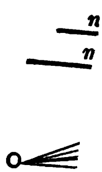


Now, since $\frac{a'b'}{AB}$

By alternation

That is, the ratic
Therefore p can
In other words,
mining ratio being

301. CONSTRU
given polygon, hav



GIVEN

TO CONSTRUCT—

of similitude be

From any point O draw lines to all the vertices A, B, C, D, E.

Construct OA' a fourth proportional to m, n, and OA. § 282

Likewise find B', C', D', E', so that:

$$\frac{m}{n}=\frac{OA}{OA'}=\frac{OB}{OB'}=\frac{OC}{OC'}=\text{etc.}$$

Then the polygons $ABCDE$ and $A'B'C'D'E'$ are similar, and their ratio of similitude is $\frac{m}{n}$. § 298

Q. E. F.

302. *Exercise.*—To draw a polygon similar to a given polygon, having a given line as a side homologous to a given side of the given polygon.

Hint.—Find the ratio of similitude. Then by § 301 construct a polygon similar to the given polygon having this ratio of similitude. Lastly, upon the given line as a side draw a polygon having its angles and sides equal to those of the second polygon.

303. *Def.*—A **diagonal** of a polygon is a straight line joining two vertices not in the same side.

304. *Exercise.*—In two similar polygons, homologous diagonals have the same ratio as any two homologous sides.

Hint.—Place the polygons in a radial position.

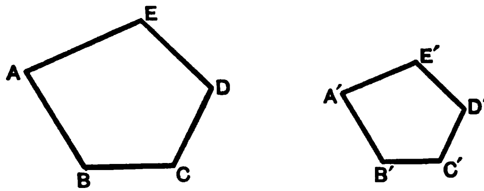
305. *Exercise.*—In two similar polygons, the straight lines joining the middle points of any two pairs of homologous sides are proportional to the sides.

306. *Exercise.*—State and prove a general proposition which includes § 305 as a special case.

307. *Def.*—The **perimeter** of a polygon is the sum of its sides.

PROPOSITION XIII. THEOREM

308. *The perimeters of two similar polygons have the same ratio as any two homologous sides.*



GIVEN—the perimeters P and P' of the two polygons $ABCDE$ and $A'B'C'D'E'$.

TO PROVE $$\frac{P}{P'}=\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}=\text{etc.}$$

Since the two polygons are similar, we have

$$\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}=\text{etc.} \qquad \S\ 274$$

Then $$\frac{AB+BC+CD+\text{etc.}}{A'B'+B'C'+C'D'+\text{etc.}}=\frac{AB}{A'B'}=\frac{BC}{B'C'}=\text{etc.} \qquad \S\ 265$$

That is, $$\frac{P}{P'}=\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}=\text{etc.}$$

Q. E. D.

309. *Remark.*—A pantograph* is a machine for drawing a plane figure similar to a given plane figure.

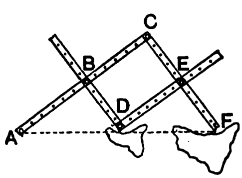


FIG. 1

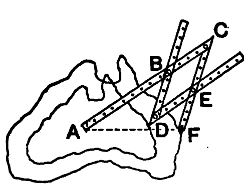


FIG. 2

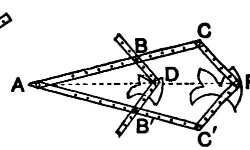


FIG. 3

* The pantograph was invented in 1603 by Christopher Scheiner. It is very [illegible] drawings.

The pantograph, shown in Figs. 1 and 2, consists of four bars, parallel in pairs and jointed at B, C, D, and E. At D and F are pencils and A turns upon a fixed pivot. BD and DE may be so adjusted as to form a parallelogram $BCED$ cutting AC and CF in any required ratio $\frac{AB}{AC}=\frac{CE}{CF}$.

Then (see § 310) D will always be in the same straight line with A and F and the ratio $\frac{AD}{AF}$ will remain constant and equal to the given ratio $\frac{AB}{AC}$.

Hence, if the pencil F traces a given figure, the pencil D will trace a similar figure, the ratio of similitude being the fixed ratio $\frac{AD}{AF}$.

In Fig. 3 the principle is similar; as also in Fig. 4, where the two figures are on opposite sides of A.

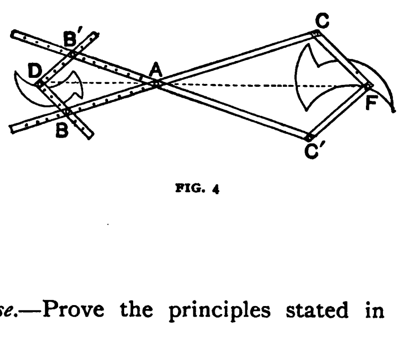


FIG. 4

310. *Exercise.*—Prove the principles stated in § 309, viz., that A, D, F remain always in the same straight line, and that $\frac{AD}{AF}$ remains constant and equal to $\frac{AB}{AC}$.

Hint.—In $\frac{AB}{AC}=\frac{CE}{CF}$ substitute BD for CE and prove the triangles ABD and ACF similar.

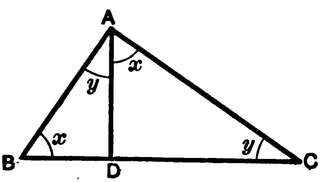
PROPOSITION XIV. THEOREM

311. *In a right triangle, if a perpendicular is drawn from the vertex of the right angle to the hypotenuse:*

I. *The triangles on each side of the perpendicular are similar to the whole triangle and to each other.*

II. *The perpendicular is a mean proportional between the segments of the hypotenuse.*

III. *Each side about the right angle is a mean proportional between the hypotenuse and the adjacent segment.*



GIVEN—the right triangle ABC and the perpendicular AD from the vertex of the right angle A on BC.

I. TO PROVE—the triangles DBA, DAC, and ABC similar to each other.

The right triangles DBA and ABC each have the angle B common; hence they are mutually equiangular. § 61

Also, the right triangles DAC and ABC, having the angle C common, are mutually equiangular. § 61

Hence the three triangles DBA, DAC, and ABC are mutually equiangular.

They are therefore similar. § 275

Q. E. D.

II. TO PROVE—AD a mean proportional between DC and BD.

Since the two right triangles DBA and DAC are similar, their homologous sides (that is, the sides opposite equal angles) are proportional. § 274

Hence BD, opposite y in triangle DBA : AD, opposite y in DAC : : AD, opposite x in first : DC, opposite x in second.

That is, AD is a mean proportional between BD and DC. § 281

Q. E. D.

III. TO PROVE—AB a mean proportional between BC and BD.

In the similar triangles ABC and DBA.

BC, opposite right angle in the large triangle : BA, opposite right angle in small : : BA, opposite y in first : BD, opposite y in second. § 274

That is, BA is a mean proportional between BC and BD.

In like manner it may be shown that AC is a mean proportional between BC and DC. Q. E. D.

312. COR. I. From II. of the preceding proposition

we have $\overline{AD}^2 = BD \times DC$, (1) § 250

and from III., $\overline{BA}^2 = BC \times BD$, (2)

and $\overline{AC}^2 = BC \times DC$. (3)

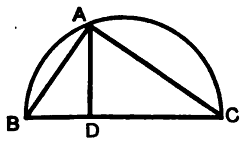
313. COR. II. Dividing (2) by (3)

$$\frac{\overline{BA}^2}{\overline{AC}^2} = \frac{BD}{DC}.$$

Hence, *in a right triangle, the squares of the sides about the right angle are proportional to the segments of the hypotenuse made by a perpendicular let fall from the vertex of the right angle.*

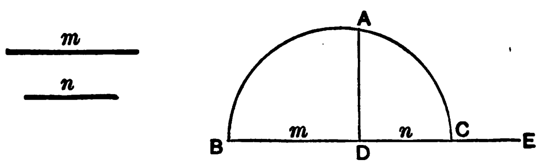
314. *Remark.*—By $\overline{AD}^2$ is understood the square of the numerical measure of AD.

315. COR. III. *If from a point A in the circumference of a circle chords AB and AC be drawn to the extremities of a diameter BC, and AD be drawn from A perpendicular to BC,*



AD will be a mean proportional between BD and DC; AB will be a mean proportional between BC and BD; and AC will be a mean proportional between BC and DC.

316. CONSTRUCTION. *To find a mean proportional between two given lines, m and n.*



On the indefinite straight line BE lay off $BD=m$ and $DC=n$.

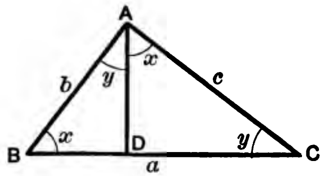
On BC as a diameter describe a semicircle.

At D erect DA perpendicular to BC, to meet the semicircle.

DA will be a mean proportional between m and n. § 315.

PROPOSITION XV. THEOREM

317. *The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.**



GIVEN—the right triangle ABC right angled at A, with sides a, b, c.

TO PROVE $b^2+c^2=a^2$.

Draw AD perpendicular to the hypotenuse BC.

Then $\left.\begin{aligned} b^2&=a\times BD \\ c^2&=a\times DC \end{aligned}\right\}$ § 312

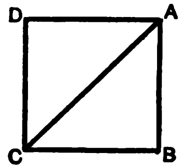
Adding $b^2+c^2=a\times(BD+DC)=a\times a$. Ax. 2

Or $b^2+c^2=a^2$. Q. E. D.

318. COR. I. *The square of either side about the right angle is equal to the difference of the squares of the other two sides.*

* This proposition was first discovered by Pythagoras in the form given in Book IV., Proposition XI. But the Egyptians are supposed to have known as early as 2000 B.C. how to make a right angle by stretching around three pegs a cord measured off into 3, 4, and 5 units. The ancient Hindoos and Chinese also used this method. It is doubtful, however, whether the fact that $3^2+4^2=5^2$ was ever observed by them. It may be noted that essentially this method of forming a right angle is still used by carpenters. Sticks of 6 feet and 8 feet form two sides, and a "ten-foot pole" completes the triangle.

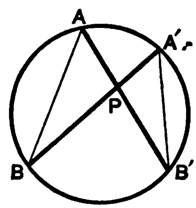
319. COR. II. *The diagonal of a square is equal to the side multiplied by the square root of two.*



OUTLINE PROOF: $AC = \sqrt{\overline{AB}^2 + \overline{BC}^2} = \sqrt{2\overline{AB}^2} = AB\sqrt{2}.$

PROPOSITION XVI. THEOREM

320. *If through a fixed point within a circle two chords are drawn, the product of the two segments of one is equal to the product of the two segments of the other.*



GIVEN—P, a fixed point in a circle, and AB' and $A'B$ any two chords drawn through P.

TO PROVE $PA \times PB' = PB \times PA'$.

Join AB and $A'B'$.

In triangles APB, $A'PB'$ angles at P are equal. § 30

[Being vertical.]

Also the angles at A and A' are equal. § 197

[Being inscribed in the same segment.]

Therefore PA, opposite $B : PA'$, opposite $B' :: PB$, opposite $A : PB'$, opposite A'. § 274

Whence $PA \times PB' = PB \times PA'$. § 250

Q. E. D.

PROPOSITION XVII. THEOREM

321. *If from a point without a circle a tangent and a secant be drawn, the tangent is a mean proportional between the whole secant and its external segment.*

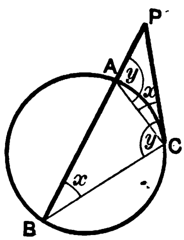


FIG. 1

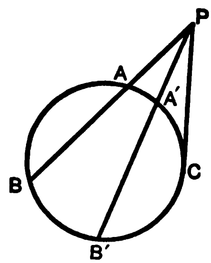


FIG. 2

GIVEN—a fixed point P outside of a circle, PC a tangent, and PB a secant (Fig. 1).

TO PROVE $\dfrac{PB}{PC} = \dfrac{PC}{PA}$.

Join AC and BC. The triangles PAC and PCB have the angle at P common, and the angles PCA and PBC (both marked x) equal, each being measured by one-half the arc AC. §§ 197, 205

Therefore the triangles are similar. § 276

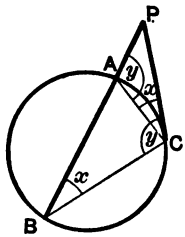


FIG. 1

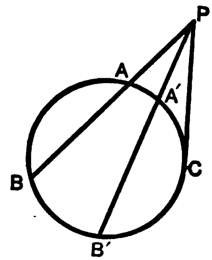


FIG. 2

Hence PB, opposite y in large triangle : PC, opposite y in small : : PC, opposite x in large : PA, opposite x in small.

Q. E. D.

322. COR. Hence, in Fig. 2,

$$PB \times PA = \overline{PC}^2,$$

and $$PB' \times PA' = PC^2.$$

Therefore $$PB' \times PA' = PB \times PA. \qquad \text{Ax. 1}$$

Hence, *if from a point without a circle two secants be drawn, the product of one secant and its external segment is equal to the product of the other and its external segment.*

323. *Exercise.*—Prove § 322 by drawing $A'B$ and AB'.

324. *Def.*—The **projection** of a straight line AB, upon another straight line MN, is the portion of MN included between the perpendiculars let fall from the extremities of AB upon the line MN.

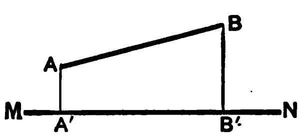


FIG. 1

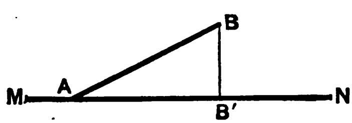


FIG. 2

In Fig. 1 $A'B'$ is the projection of AB. In Fig. 2, where one extremity

PROPOSITION XVIII. THEOREM

325. *In any triangle, the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides, minus twice the product of one of these sides and the projection of the other side upon it.*

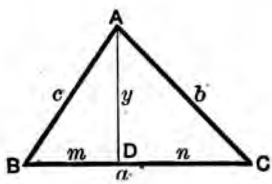


FIG. 1

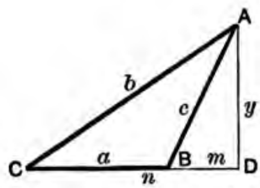


FIG. 2

GIVEN the triangle ABC and C, an acute angle.

Draw AD perpendicular to CB or CB produced, making CD the projection of AC on CB, and call $AB=c$; $AC=b$; $BC=a$; $AD=y$; $BD=m$; $CD=n$.

TO PROVE $c^2=a^2+b^2-2an$.

In the right triangle ABD.

$$c^2=m^2+y^2. \quad (1)$$ § 317

In Fig. 1, $m=a-n$; and in Fig. 2, $m=n-a$.

In *both* cases $m^2=a^2-2an+n^2$.

Substituting this value in (1),

$$c^2=a^2-2an+n^2+y^2. \quad (2)$$

But in the triangle ACD, $n^2+y^2=b^2$. § 317

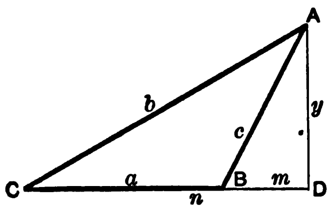
Substituting this value in (2),

$$c^2=a^2+b^2-2an.$$ Q. E. D.

SUMMARY: $c^2=m^2+y^2=a^2-2an+n^2+y^2=a^2-2an+b^2$.

PROPOSITION XIX. THEOREM

326. *In an obtuse-angled triangle the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides, plus twice the product of one of these sides and the projection of the other side upon it.*



GIVEN—the obtuse-angled triangle ABC with B the obtuse angle.

Draw AD perpendicular to CB produced, making BD the projection of AB on CB, and call $AB=c$; $AC=b$; $BC=a$; $AD=y$; $BD=m$; $CD=n$.

TO PROVE $b^2=a^2+c^2+2am$.

In the right triangle ACD

$$b^2=n^2+y^2. \quad (1) \qquad \S\ 317$$

But $n=a+m$.

And $n^2=a^2+2am+m^2$.

Substituting this value of n^2 in (1),

$$b^2=a^2+2am+m^2+y^2. \quad (2)$$

But in the triangle ABD, $m^2+y^2=c^2$. § 317

Substituting this value in (2),

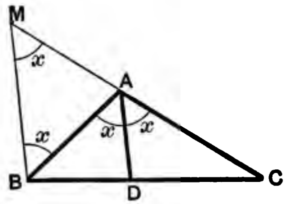
$$b^2=a^2+c^2+2am.$$

Q. E. D.

SUMMARY: $b^2=n^2+y^2=a^2+2am+m^2+y^2=a^2+2am+c^2$.

PROPOSITION XX. THEOREM

327. *The bisector of an angle of a triangle divides the opposite side into segments which are proportional to the other two sides.*



GIVEN—in the triangle ABC, AD the bisector of the angle A.

TO PROVE $$\frac{DC}{DB}=\frac{AC}{AB}.$$

Draw BM parallel to AD and meeting AC produced at M.

Then in the triangle BMC, since AD is parallel to BM,

$$\frac{DC}{DB}=\frac{AC}{AM}. \quad (1) \qquad \S\ 271$$

Also, since AD is parallel to MB,

angle $M = DAC$. § 49

[Being corresponding angles of parallel lines.]

And angle $MBA = BAD$. § 48

[Being alt.-int. angles of parallel lines.]

But angle $DAC = BAD$. Hyp.

Therefore angle $M = MBA$. Ax. 1

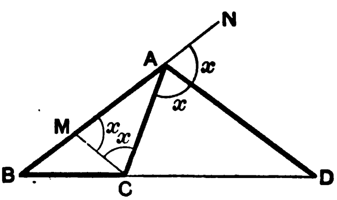
And $AM = AB$. § 77

Substituting in (1), $$\frac{DC}{DB}=\frac{AC}{AB}. \qquad \text{Q. E. D.}$$

328. COR. Conversely, *if AD divides BC into two segments which are proportional to the adjacent sides, it bisects the angle BAC.*

PROPOSITION XXI. THEOREM

329. *The bisector of an exterior angle of a triangle meets the opposite side produced in a point whose distances from the extremities of that side are proportional to the other two sides.*



GIVEN—in the triangle ABC, AD the bisector of the exterior angle CAN.

TO PROVE $$\frac{DB}{DC}=\frac{AB}{AC}.$$

Draw CM parallel to AD, meeting AB at M.

Then in the triangle BAD, since CM is parallel to AD,

$$\frac{DB}{DC}=\frac{AB}{AM}. \quad (1) \qquad \S\ 272$$

Also,	since CM is parallel to AD,	
	angle $AMC=NAD$.	§ 49
And	angle $ACM=CAD$.	§ 48
But	angle $NAD=CAD$.	Hyp.
Therefore	angle $AMC=ACM$.	Ax. 1
And	$AM=AC$.	§ 77

Substituting in (1), $$\frac{DB}{DC}=\frac{AB}{AC}.$$ Q. E. D.

330. COR. Conversely, *if AD meets BC produced so that* $\dfrac{DB}{DC}=\dfrac{AB}{AC}$, *then it bisects the angle CAN.*

331. *Defs.*—The line AB is divided **internally** at C, when this point is between the extremities of the line; CA and CB are the segments into which it is divided.

AB is divided **externally** at C', when this point is on the line produced. The segments are $C'A$ and $C'B$.

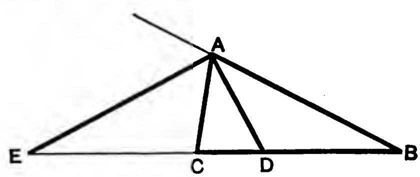
In each case the segments are the distances from the point of division to the extremities of the line. The line is the *sum* of the internal segments, and the *difference* of the external segments.

332. A line is divided **harmonically**, when it is divided internally and externally in the same ratio.

Thus, if $\dfrac{CA}{CB}=\dfrac{C'A}{C'B}$, then AB is divided harmonically at C and C'.

333. *Exercise.*—Prove that the bisectors of the interior and exterior angles at one of the vertices of a triangle divide the opposite side harmonically (see figure below).

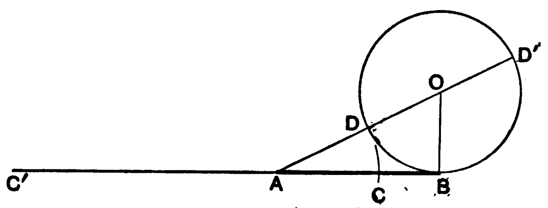
334. *Exercise.*—If AD and AE bisect the angles at A, prove also that ED is divided harmonically at C and B.



Hint.—Alternate the proportion found in § 333.

335. *Def.*—A straight line is divided in **extreme and mean ratio** when one of its segments is a mean proportional between the whole line and the other segment.

336. CONSTRUCTION. *To divide a given straight line in extreme and mean ratio.*



GIVEN the straight line AB.

REQUIRED to divide it in extreme and mean ratio.

At B draw the perpendicular BO equal to one half AB.

With the centre O and radius OB describe a circumference, and draw AO, cutting the circumference in D and D'.

On AB lay off $AC=AD$, and extend BA to C', making $AC'=AD'$.

Then AB is divided in extreme and mean ratio, internally at C, and externally at C'.

I. $$\frac{AD'}{AB}=\frac{AB}{AD}. \quad (1) \qquad \S\ 321$$

By division and inversion

$$\frac{AB}{AD'-AB}=\frac{AD}{AB-AD}. \quad (2) \qquad \S\S\ 254,\ 259$$

But $AB=2OB=DD'$, and $AD=AC$. Cons.

Therefore,

$AD'-AB=AD'-DD'=AD=AC$, and $AB-AD=BC$.

Substituting these values in (2),

$$\frac{AB}{AC}=\frac{AC}{BC}.$$

Hence AB is divided internally at C in extreme and mean ratio. Q. E. F.

II. By composition and inversion of (1),

$$\frac{AD'}{AD'+AB}=\frac{AB}{AB+AD}. \quad (3) \qquad \text{§§ 254, 257}$$

But $AD'=AC'$, and $AB=DD'$.

Therefore $AD'+AB=AC'+AB=BC'$,

And $AB+AD=DD'+AD=AD'=AC'$.

Substituting these values in (3),

we obtain $$\frac{AB}{AC'}=\frac{AC'}{BC'}.$$

Hence AB is divided externally at C' in extreme and mean ratio. Q. E. F.

337. *Remark.*—AC and AC' may be computed in terms of AB as follows:

$$AC=AD=AO-OD=AO-\frac{AB}{2}. \quad (1)$$

Likewise $AC'=AD'=AO+OD'=AO+\frac{AB}{2}. \quad (2)$

But $\overline{AO}^2=\overline{AB}^2+\left(\frac{AB}{2}\right)^2=\overline{AB}^2+\overline{AB}^2.\frac{1}{4}=\overline{AB}^2.\frac{5}{4}.$ § 317

Whence, extracting the square root,

$$AO=AB.\frac{\sqrt{5}}{2}.$$

Substituting in (1) and (2),

$$AC=AB.\frac{\sqrt{5}}{2}-\frac{AB}{2}=AB.\frac{\sqrt{5}-1}{2}.$$

And $$AC'=AB.\frac{\sqrt{5}}{2}+\frac{AB}{2}=AB.\frac{\sqrt{5}+1}{2}.$$

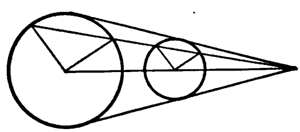
PROBLEMS OF DEMONSTRATION

338. *Exercise.*—The point of intersection of the internal tangents to two circles divides the line of centres internally into parts whose ratio equals the ratio of the radii.

339. *Exercise.*—The point of intersection of the external tangents to two circles divides the line of centres externally into parts whose ratio equals the ratio of the radii.

340. *Exercise.*—The points of intersection of the internal and external tangents to two circles divide the line of centres harmonically.

341. *Exercise.*—If through the centres of two circles two parallel radii are drawn in the same direction, the straight line joining their extremities will pass through the intersection of the external tangents.



342. *Exercise.*—If through the centres of two circles two parallel radii are drawn in opposite directions, the straight line joining their extremities will pass through the intersection of the internal tangents.

343. *Exercise.*—If through the intersection of the external or of the internal tangents to two circles a secant is drawn, the radii to the points of intersection will be parallel in pairs.

344. *Exercise.*—Give methods for drawing the common tangents to two circles depending on §§ 341, 342.

345. *Exercise.*—A triangle ABC is inscribed in a circle to which a second circle is externally tangent at A. If AB and AC are produced till they meet the second circumference at M and N, the triangles ABC and AMN are similar. §§ 205, 275

346. *Exercise*—The perpendiculars from any two vertices of a triangle on the opposite sides are inversely proportional to those sides. § 276

347. *Exercise.*—If two circles are tangent internally, all chords of the greater drawn from the point of contact are divided proportionally by the circumference of the smaller.

Hint.—Apply §§ 202, 225, 276.

348. *Exercise.*—If from P, a point in a circumference, any chords, PA, PB, PC, are drawn, and these chords are cut in a, b, c, respectively, by any straight line parallel to the tangent at P, then $PA \times Pa = PB \times Pb = PC \times Pc$.

Hint.—Let one chord pass through centre. Join its extremity to any other chord and apply §§ 202, 276.

349. *Exercise.*—On a common base AB are two triangles, ABC and ABC', whose vertices C and C' lie in a straight line parallel to AB. If a second parallel to AB cuts AC and BC in M and N, and AC' and BC' in M' and N', then $MN = M'N'$. § 275

350. *Exercise.*—If at the extremities of BC, the hypotenuse of a right triangle ABC, perpendiculars to the hypotenuse are drawn intersecting AB produced in M and AC produced in N, then $\frac{AB}{AN} = \frac{AM}{AC}$.

351. *Exercise.*—The difference of the squares of two sides of any triangle is equal to the difference of the squares of the projections of these sides on the third side. § 317

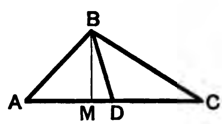
352. *Exercise.*—If from one of the acute angles of a right-angled triangle a straight line be drawn bisecting the opposite side, the square of that line will be less than the square of the hypotenuse by three times the square of half the side bisected.

353. *Exercise.*—If two circles intersect each other, the tangents drawn from any point of their common chord produced are equal. § 321

354. *Exercise.*—If two circles intersect each other, their common chord if produced will bisect their common tangent. § 321

355. *Exercise.*—I. The sum of the squares of two sides of a triangle is equal to twice the square of half the third side, plus twice the square of the median drawn to the third side.

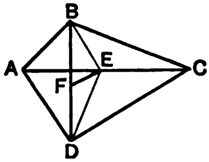
II. The difference of the squares of two sides of a triangle is equal to twice the product of the third side by the projection of the median upon the third side.



Hint.—The median BD divides ABC into two triangles, one acute angled and the other obtuse angled (provided AB and BC are not equal).

Apply §§ 325, 326.

356. *Exercise.* — In any quadrilateral the sum of the squares of the four sides is equal to the sum of the squares of the diagonals plus four times the square of the line joining the middle points of the diagonals.

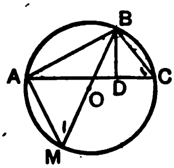


Hint.—Apply § 355, I. to the triangles ABC, ADC, and BED, and combine equations thus obtained.

357. *Exercise.*—The product of two sides of a triangle is equal to the product of the diameter of the circumscribed circle and the altitude upon the third side.

$\frac{AB}{BD} = \frac{BM}{BC}$

$AB \times BC = BM \times BD.$



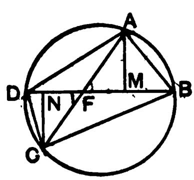
$AB \times BC = BM \times BD.$

Hint.—Let ABC be the triangle. Draw the altitude BD and the diameter BM. Prove the triangles BAM and BDC similar. §§ 201, 202, 276

358. *Exercise.*—In an inscribed quadrilateral, $ABCD$, if F is the intersection of the diagonals AC and BD, then

$$\frac{AB \times AD}{CB \times CD} = \frac{AF}{FC}.$$

$\frac{AM}{CN} = \frac{AF}{CF}$



$AB \times AD = X \times AM.$

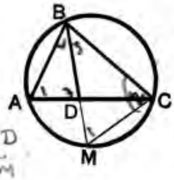
$CB \times CD = CN \cdot X.$

Hint.—In the triangles ABD and CBD, draw the altitudes AM and CN and apply § 357. Then compare triangles AFM and CFN.

359. *Exercise.*—The product of two sides of a triangle is equal to the square of the bisector of their included angle plus the product of the segments of the third side formed by the bisector.

$AB \times BC = \overline{BC}^2 + AD \times DC$

$BD \times DM = AD \times DC.$



$\frac{BD}{BC} = \frac{AB}{CM} = \frac{AD}{BM}$

$AB \times BC = BD \times CM.$

Hint.—Circumscribe a circle about ABC and produce the bisector to cut the circumference in M. Prove the triangles ABD and MBC similar. Apply § 320.

PROBLEMS OF CONSTRUCTION

360. *Exercise.*—To produce a given straight line MN to a point X, such that $MN : MX = 3 : 7$.

361. *Exercise.*—To construct two straight lines having given their sum and ratio.

362. *Exercise.*—Having given the lesser segment of a straight line divided in extreme and mean ratio, to construct the whole line.

363. *Exercise.*—To construct a triangle having a given perimeter and similar to a given triangle.

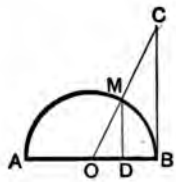
364. *Exercise.*—To construct a right triangle having given an acute angle and the perimeter.

365. *Exercise.*—To divide one side of a given triangle into segments proportional to the other two sides.

366. *Exercise.*—In a given circle to inscribe a triangle similar to a given triangle.

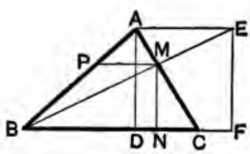
367. *Exercise.*—About a given circle to circumscribe a triangle similar to a given triangle.

368. *Exercise.*—To inscribe a square in a semicircle.



Hint.—At B draw CB equal and perpendicular to the diameter. Join OC cutting the circumference in M, and draw MD parallel to CB. Prove MD the side of the required square by § 275.

369. *Exercise.*—To inscribe a square in a given triangle.

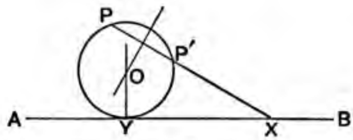


Hint.—On the altitude AD construct the square $ADFE$ and draw BE cutting the side AC at M. From M draw MN and MP parallel to EF and AE respectively. Prove these lines equal and sides of the required square.

370. *Exercise.*—To inscribe in a given triangle a rectangle similar to a given rectangle.

371. *Exercise.*—To inscribe in a given triangle a parallelogram similar to a given parallelogram.

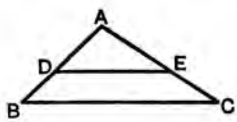
372. *Exercise.*—To construct a circumference which shall pass through two given points and be tangent to a given straight line.



Hint.—Let AB be the given line, P and P' the points. If the straight line PP' is parallel to AB, the solution is simple. If PP' is not parallel to AB, it will cut it at some point X, and the distance from X to Y, the required point of tangency, may be determined by § 321.

PROBLEMS FOR COMPUTATION

373. (1.) In the triangle ABC, DE is drawn parallel to BC. If $\frac{AD}{DB}=\frac{4}{3}$, $BC=56$, and $AE=24$, find AC and DE.



(2.) The sides of a triangle are 3, 5, and 7. In a similar triangle the side homologous to 5 is equal to 65. Find the other two sides of the second triangle.

(3.) The shadow cast upon level ground by a certain church steeple is 27 yds. long, while at the same time that of a vertical rod 5 ft. high is 3 ft. long. Find the height of the steeple.

(4.) The footpaths on the opposite sides of a street are 30 ft. apart. On one of them a bicycle rider is moving uniformly at the rate of 15 miles per hour. If a man on the other side, walking in the opposite direction, so regulates his pace that a tree 5 ft. from his path continually hides him from the rider, does he walk uniformly, and, if so, at what rate does he walk?

(5.) If from the top of a telegraph-pole standing upon the brink of a stream 23 m. wide a wire 30 m. long reaches to the opposite side of the stream, how high is the pole?

(6.) Given the two perpendicular sides of a right triangle equal to 8 and 6 in. respectively to compute the length of the perpendicular from the vertex of the right angle to the hypotenuse.

(7.) If in a right triangle the two perpendicular sides are a and b, compute the altitude upon the hypotenuse.

(8.) If, in the above example, $a = 137.53$ dkm., and $b = 213.19$ m., find the altitude.

(9.) If in a right triangle one of the sides about the right angle is double the other, what is the ratio of the segments of the hypotenuse formed by the altitude upon the hypotenuse?

(10.) There are two telegraph-poles standing upon the same level in a city street, one 59 ft. high, the other 45 ft. high, while between them, and in a straight line with their bases, is a hitching-post 3 ft. high. If the distance from the top of the post to the top of the higher pole is 100 ft., and from the top of the post to that of the lower pole 80 ft., how far apart are the poles?

(11.) If the chord of an arc is 720 ft. and the chord of its half is 369 ft., what is the diameter of the circle?

(12.) A chord of a circle is divided into two segments of 73.162 dcm. and 96.758 dcm. respectively by another chord, one of whose segments is 3.1527 m. What is the length of the second chord?

(13.) If a chord of a circle is cut by another chord into two segments, a and b, and one segment of the second chord is equal to c, find the other segment.

(14.) If from a point without a circle two secants are drawn whose external segments are 8 in. and 7 in., while the internal segment of the latter is 17 in., what is the length of the internal segment of the former?

(15.) From a point without a circle are drawn a tangent and a secant, the secant passing through the centre. If the length of the tangent is a, and the external segment of the secant is b, find the radius of the circle.

(16.) In a triangle whose sides are respectively 25.136 cm., 31.298 cm., and 37.563 cm. in length, find the segments of the longest side formed by the bisector of the opposite angle.

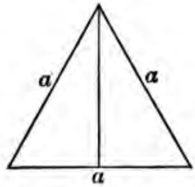
(17.) In a triangle whose sides are a, b, and c, find the segments of the side b formed by the bisector of the opposite angle.

(18.) If the base of an isosceles triangle is 60 cm., and each of its sides is 50 cm., find the length of its altitude in inches.

(19.) If the base of an isosceles triangle is b, and its altitude h, find the sides.

(20.) Find the altitude of an equilateral triangle whose side is 5 in.

(21.) Show that, if a is the side of an equilateral triangle, the altitude is $\frac{1}{2}a\sqrt{3}$.



(22.) Find in feet the side of an equilateral triangle having an altitude of 793.57 m.

(23.) Show that, in a right triangle, one of whose acute angles is 30°, and whose hypotenuse is a, the side opposite 30° is $\frac{1}{2}a$, and the other side is $\frac{1}{2}a\sqrt{3}$.

(24.) One acute angle of a right triangle is 30° and the hypotenuse is 4.3791 cm. Find the other sides.

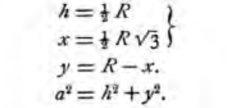
(25.) Find the side of an isosceles right triangle whose hypotenuse is 3 ft.

(26.) If a is the hypotenuse of an isosceles right triangle, the side is $\frac{1}{2}a\sqrt{2}$.

(27.) Find the side of an isosceles right triangle whose hypotenuse is 32.174 dkm.

(28.) Find the base of an isosceles triangle whose side is 4 ft. and whose vertex angle is 30°.

(29.) If one of the equal sides of an isosceles triangle is R and the vertex angle is 30°, show that the base is $R\sqrt{2-\sqrt{3}}$.



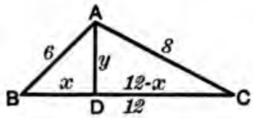
Hint.

$$\left.\begin{array}{l} h = \frac{1}{2}R \\ x = \frac{1}{2}R\sqrt{3} \end{array}\right\} \qquad \S\ 373(23)$$

$$y = R - x.$$

$$a^2 = h^2 + y^2.$$

(30.) Having given a triangle whose sides are 6, 8, and 12, find its altitude upon the side 12.



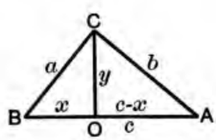
Solution.—In the triangle ABD, $y^2+x^2=36$. § 317
In the triangle ADC, $y^2+(12-x)^2=64$.
Combine the two equations and eliminate y.

$$y^2+x^2=36 \quad (1)$$
$$y^2-24x+x^2=-80 \quad (2)$$
$$24x=116$$
$$x=\tfrac{29}{6}=4\tfrac{5}{6}.$$

Substituting this value in (1),

$$y^2+(\tfrac{29}{6})^2=36$$
$$36y^2=455$$
$$6y=\sqrt{455}=21.33+$$
$$y=3.55+$$

(31.) In a triangle whose sides are a, b, and c, find the three altitudes.



Solution.—In the triangle CBO, $x^2+y^2=a^2$. (1)
In the triangle CAO, $(c-x)^2+y^2=b^2$. (2) § 317

Simplifying and combining,

$$x^2+y^2=a^2$$
$$x^2-2cx+y^2=b^2-c^2$$
$$2cx=a^2-b^2+c^2$$
$$x=\frac{a^2+c^2-b^2}{2c}.$$

Substituting value of x in (1),

$$\left(\frac{a^2+c^2-b^2}{2c}\right)^2+y^2=a^2$$
$$y^2=a^2-\left(\frac{a^2+c^2-b^2}{2c}\right)^2$$
$$y=\sqrt{a^2-\left(\frac{a^2+c^2-b^2}{2c}\right)^2}.$$

This result may be factored and arranged for logarithmic computation as follows:

$$y=\sqrt{a^2-\left(\frac{a^2+c^2-b^2}{2c}\right)^2}=\sqrt{\left(a+\frac{a^2+c^2-b^2}{2c}\right)\left(a-\frac{a^2+c^2-b^2}{2c}\right)}$$
$$=\sqrt{\left(\frac{2ac+a^2+c^2-b^2}{2c}\right)\left(\frac{2ac-a^2-c^2+b^2}{2c}\right)}$$
$$=\sqrt{\frac{1}{c^2}\left(\frac{(a+c)^2-b^2}{2}\right)\left(\frac{b^2-(a-c)^2}{2}\right)}.$$

Multiplying each fraction by $\frac{2}{2}$, and factoring,

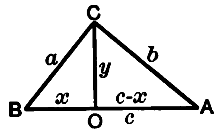
$$y=\sqrt{\frac{4}{c^2}\left(\frac{a+b+c}{2}\right)\left(\frac{a+c-b}{2}\right)\left(\frac{a+b-c}{2}\right)\left(\frac{b+c-a}{2}\right)}.$$

Let $\qquad \frac{a+b+c}{2}=s.$

Then $\qquad \frac{a+b+c}{2}-b=s-b.$ $\qquad$ Ax. 3

Whence $\qquad \frac{a+c-b}{2}=s-b.$

In same manner $\frac{a+b-c}{2}=s-c$, and $\frac{b+c-a}{2}=s-a.$



Substituting these values under radical and extracting root,

$$y=\frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}.$$

The other altitudes are

$$\frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$$

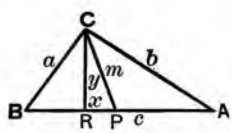
and $$\frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}.$$

(32.) Having given the sides of a triangle equal to 375.49, 289.63, and 231.19, find its three altitudes.

(33.) If the sides of a triangle are 27.931 m., 2175.4 cm., and 296.53 dcm., what are the lengths in feet of (1) the altitude upon the greatest side, and (2) the segments into which it divides that side?

Hint.—After finding the altitude, the segments can easily be found by logarithms, since (§ 318) $x=\sqrt{a^2-y^2}=\sqrt{(a-y)(a+y)}$.

(34.) Compute the medians of a triangle whose sides are a, b, and c.



Solution.—In the triangle CRP, $m^2=x^2+y^2$. (1)

In the triangle CRA, $y^2+\left(\frac{c}{2}+x\right)^2=b^2$. (2)

In the triangle CBR, $y^2+\left(\frac{c}{2}-x\right)^2=a^2$. (3)

§ 317

Simplifying, $$y^2+\frac{c^2}{4}+cx+x^2=b^2. \quad (2)$$

$$y^2+\frac{c^2}{4}-cx+x^2=a^2. \quad (3)$$

Adding, $$2y^2+\frac{c^2}{2}+2x^2=a^2+b^2.$$

Transposing, $$2(x^2+y^2)=a^2+b^2-\frac{c^2}{2}=\frac{2(a^2+b^2)-c^2}{2}$$

$$x^2+y^2=\frac{2(a^2+b^2)-c^2}{4}.$$

But $$x^2+y^2=m^2. \quad (1)$$

Therefore $$m^2=\frac{2(a^2+b^2)-c^2}{4}.$$

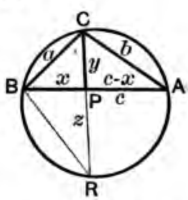
$$\boldsymbol{m=\tfrac{1}{2}\sqrt{2(a^2+b^2)-c^2}.}$$

The other medians are $\frac{1}{2}\sqrt{2(b^2+c^2)-a^2}$ and $\frac{1}{2}\sqrt{2(c^2+a^2)-b^2}$.

(35.) Having given the three sides of a triangle equal to 3, 5, and 7, find its three medians.

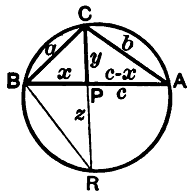
(36.) If two sides and one of the diagonals of a parallelogram are respectively 24, 31, and 28, what is the length of the other diagonal?

(37.) In a triangle whose sides are a, b, and c, compute the bisector of the angle opposite c.



Solution.—Circumscribe a circle about the triangle, produce the bisector to meet the circumference, and draw BR. Then, in the triangles BCR and CPA, the angle R equals the angle A and angle BCR equals the angle PCA. § 201

A



Therefore $\dfrac{y+z}{b}=\dfrac{a}{y}$. § 274

Whence $y^2+yz=ab$. (1) § 250

But $\dfrac{a}{b}=\dfrac{x}{c-x}$. § 327

Whence $bx=ac-ax$, § 250

$$x=\frac{ac}{a+b}, \text{ (2); and } c-x=\frac{bc}{a+b}. \text{ (3)}$$

But $(c-x)\times x=y\times z$. § 320

Substituting values for x and $(c-x)$ from (2) and (3)

$$yz=\frac{abc^2}{(a+b)^2}. \quad (4)$$

Subtracting (4) from (1) $y^2=ab-\dfrac{abc^2}{(a+b)^2}=ab\left(1-\dfrac{c^2}{(a+b)^2}\right)$.

$$\boldsymbol{y=\sqrt{ab\left(1-\frac{c^2}{(a+b)^2}\right)}}.$$

This result may be factored and arranged for logarithmic computation as follows:

$$\sqrt{ab\left(1-\frac{c^2}{(a+b)^2}\right)}=\sqrt{ab\left(1+\frac{c}{a+b}\right)\left(1-\frac{c}{a+b}\right)}$$

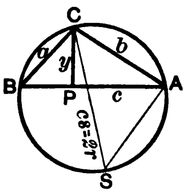
$$=\sqrt{ab\left(\frac{a+b+c}{a+b}\right)\left(\frac{a+b-c}{a+b}\right)}.$$

Multiplying both fractions by $\frac{2}{2}$, and extracting root,

$$y=\frac{2}{a+b}\sqrt{ab\left(\frac{a+b+c}{2}\right)\left(\frac{a+b-c}{2}\right)}=\boldsymbol{\frac{2}{a+b}\sqrt{abs(s-c)}}.$$

(38.) If the sides of a triangle are 219.57, 178.35, and 153.94 ft., find the length of the bisector of the angle opposite the greatest side.

(39.) If the sides of a triangle are a, b, and c, find the radius of the circumscribed circle.



Solution.—Suppose the diameter CS of the circle to be drawn from C. Draw SA and the altitude CP.

Then in the right triangles CSA and CBP the angle CAS is equal to the angle P (§ 202), and the angle S is equal to the angle B. § 201

Therefore the triangles are similar, and

$$\frac{2r}{a}=\frac{b}{y}.$$

Hence $$2ry=ab.$$

And $$r=\frac{ab}{2y}.$$

But by Problem (31) $$y=\frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}.$$

Substituting this value,

$$r=\frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}.$$

(40.) If the sides of a triangle are 125.76, 119.53, and 98.991 ft. in length, find the radius of the circumscribing circle expressed in meters.

PLANE GEOMETRY

BOOK IV

AREAS OF POLYGONS

374. *Def.*—The **area** of a surface is the ratio of that surface to another surface taken as the unit.

The unit surface may have any size or shape, but the most common and convenient unit is a square having its side equal to the unit of length, as a square inch, a square mile, etc.

375. *Def.*—**Equivalent** figures are figures having equal areas.

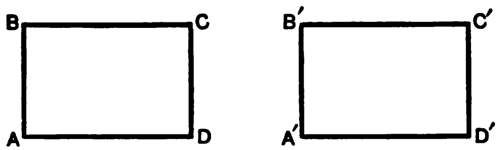
We may observe (1) figures of the same *shape* are *similar*.
(2) figures of the same *size* are *equivalent*.
(3) figures of the same *shape and size* are *equal*.

376. *Defs.*—The **bases** of a parallelogram are the side upon which it is supposed to stand and the opposite side.

The **altitude** is the perpendicular distance between the bases.

PROPOSITION I. THEOREM

377. *Two rectangles having equal bases and equal altitudes are equal.*



GIVEN—two rectangles, AC and $A'C'$, having equal bases, AD and $A'D'$, and equal altitudes, AB and $A'B'$.

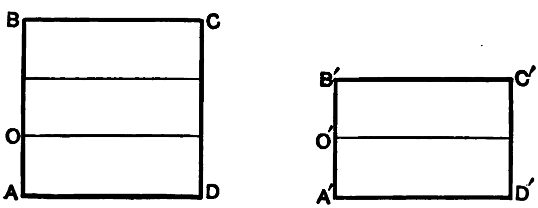
TO PROVE the rectangles equal.

Make AD coincide with its equal $A'D'$.
Then AB will take the direction of $A'B'$. § 18
And B will fall on B'. Hyp.
That is, AB will coincide with $A'B'$.
Similarly DC will coincide with $D'C'$.
And therefore BC will coincide with $B'C'$. Ax. *a*
Hence the rectangles coincide throughout and are equal. § 15

Q. E. D.

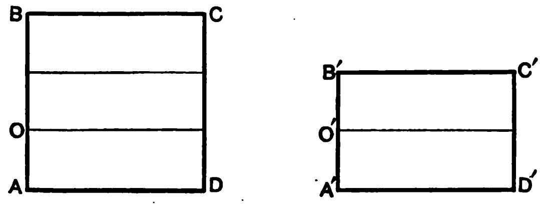
PROPOSITION II. THEOREM

378. *Two rectangles having equal bases are to each other as their altitudes.*



GIVEN—two rectangles AC and $A'C'$, having equal bases, AD and $A'D'$.

TO PROVE $\dfrac{\text{rect. } AC}{\text{rect. } A'C'} = \dfrac{AB}{A'B'}$.



CASE I. *When the altitudes, AB and A'B', are commensurable.*

Suppose AO, the common measure of the altitudes, is contained in AB three times and in $A'B'$ twice.

Then $$\frac{AB}{A'B'}=\frac{3}{2}.$$ § 180

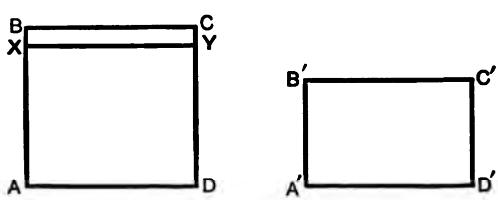
Through the several points of division draw parallels to the bases.

The rectangle AC will be divided into three rectangles and $A'C'$ into two, all five of which will be *equal.* § 377

Hence $$\frac{\text{rect. } AC}{\text{rect. } A'C'}=\frac{3}{2}.$$ § 180

Therefore $$\frac{\text{rect. } AC}{\text{rect. } A'C'}=\frac{AB}{A'B'}.$$ Ax. 1

CASE II. *When the altitudes, AB and A'B', are incommensurable.*



Suppose $A'B'$ to be divided into any number of equal parts and apply one of these parts to AB as a measure as often as it will be exactly contained.

Since AB and $A'B'$ are incommensurable, there will be a remainder XB, less than one of these parts.

Draw XY parallel to the base.

Since AX and $A'B'$ are constructed commensurable,

$$\frac{\text{rect. } AY}{\text{rect. } A'C'} = \frac{AX}{A'B'}. \qquad \text{Case I}$$

Now suppose the number of parts into which $A'B'$ is divided to be indefinitely increased.

We can thus make each part as small as we please.

But the remainder XB will always be less than one of these parts.

Therefore we can make XB less than any assigned quantity, though never zero.

That is, AX approaches AB as its limit. § 185

Likewise rect. AY approaches rect. AC as its limit.

$$\left.\begin{array}{l} \text{Hence } \dfrac{AX}{A'B'} \text{ approaches } \dfrac{AB}{A'B'} \text{ as its limit.} \\ \text{Also } \dfrac{\text{rect. } AY}{\text{rect. } A'C'}, \text{ approaches } \dfrac{\text{rect. } AC}{\text{rect. } A'C'} \text{ as its limit.} \end{array}\right\} \quad \text{§ 190}$$

But since
$$\frac{\text{rect. } AY}{\text{rect. } A'C'} = \frac{AX}{A'B'},$$

then
$$\frac{\text{rect. } AC}{\text{rect. } A'C'} = \frac{AB}{A'B'}. \qquad \text{§ 186}$$

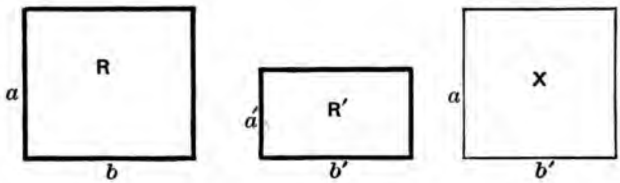
[If two variables are always equal and each approaches a limit, the limits are equal.] Q. E. D.

379. Cor. *Two rectangles having equal altitudes are to each other as their bases.*

Hint.—AD and $A'D'$ may be regarded as the altitudes, and AB and $A'B'$ as the bases.

PROPOSITION III. THEOREM

380. *Any two rectangles are to each other as the products of their bases and altitudes.*



GIVEN—any two rectangles, R and R', their bases being b and b', and altitudes a and a'.

TO PROVE $$\frac{R}{R'}=\frac{a\times b}{a'\times b'}.$$

Construct rectangle X, having the same base as R' and altitude as R.

Then $$\frac{R}{X}=\frac{b}{b'}.$$ § 379

[Two rectangles having equal altitudes are to each other as their bases.]

And $$\frac{X}{R'}=\frac{a}{a'}.$$ § 378

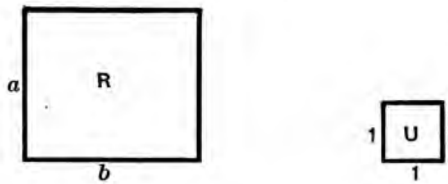
[Two rectangles having equal bases are to each other as their altitudes.]

Multiplying, $$\frac{R}{X}\times\frac{X}{R'}=\frac{b}{b'}\times\frac{a}{a'},$$

or $$\frac{R}{R'}=\frac{a\times b}{a'\times b'}.$$ Q. E. D.

PROPOSITION IV. THEOREM

381. *The area of a rectangle equals the product of its base and altitude, provided the unit of area is a square whose side is the linear unit.*



GIVEN—the rectangle R and a square U with each side a linear unit.

TO PROVE—area of $R = a \times b$, provided U is the unit of area.

$$\frac{R}{U} = \frac{a \times b}{1 \times 1} = a \times b. \qquad \S\ 380$$

[Two rectangles are to each other as the products of their bases by their altitudes.]

But $$\frac{R}{U} = \text{area of } R. \qquad \S\ 374$$

[The area of a surface is the ratio of that surface to the unit surface.]

Therefore area of $R = a \times b$,
provided U is the unit of area. Ax. 1

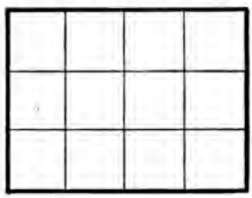
Q. E. D.

382. *Remark.*—Hereafter it is to be understood without any express proviso that we take as the unit of area a square whose side is the linear unit.

383. COR. *The area of any square equals the second power of its side.*

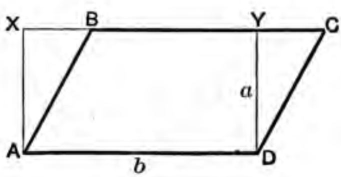
This fact is the origin of the custom of calling the second power of a number its "square."

384. *Remark.*—When the base and altitude of a rectangle each contain the linear unit an exact number of times, Proposition IV. becomes evident to the eye. Thus, if the base contain four and the altitude three linear units, the figure may be divided into twelve unit squares.



PROPOSITION V. THEOREM

385. *The area of a parallelogram equals the product of its base and altitude.*



GIVEN—the parallelogram $ABCD$, with base b and altitude a.

TO PROVE the area of $ABCD = a \times b$.

Draw AX and DY perpendiculars between the parallels AD and BC.

Then $ADYX$ is a rectangle, having the same base and altitude as the parallelogram.

Right triangle AXB = right triangle DYC. (Why?)

Take away the right triangle DYC from the whole figure, and we have left the rectangle $ADYX$.

Take away the right triangle AXB from the whole figure, and we have left the parallelogram $ABCD$.

Therefore area $ADYX$ = area $ABCD$. Ax. 3

But area $ADYX = a \times b$. § 381

[The area of a rectangle equals the product of its base by its altitude.]

Therefore area $ABCD = a \times b$. Ax. 1

Q. E. D.

386. COR. I. *Parallelograms having equal bases and equal altitudes are equivalent.*

387. COR. II.—*Any two parallelograms are to each other as the products of their bases and altitudes.*

Hint.—Let the areas of the parallelograms be P and P', their bases b and b', and altitudes a and a'.

Then $P = ab$ and $P' = a'b'$.

And $\frac{P}{P'} = \frac{ab}{a'b'}$.

388. COR. III. *Two parallelograms having equal bases are to each other as their altitudes.*

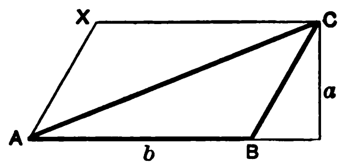
$$\left(\frac{P}{P'} = \frac{a \times b}{a' \times b} = \frac{a}{a'}.\right)$$

389. COR. IV. *Two parallelograms having equal altitudes are to each other as their bases.*

$$\left(\frac{P}{P'} = \frac{a \times b}{a \times b'} = \frac{b}{b'}.\right)$$

PROPOSITION VI. THEOREM

390. *The area of a triangle equals one-half the product of its base and altitude.*



GIVEN the triangle ABC with base b and altitude a.

TO PROVE area $ABC = \frac{1}{2} a \times b$.

From C draw CX parallel to AB.

From A draw AX parallel to BC.

Then the figure $ABCX$ is a parallelogram. § 114

and the triangle $ABC = \frac{1}{2}$ the parallelogram $ABCX$. § 116

[The diagonal of a parallelogram divides it into two equal triangles.]

But area paral. $ABCX = a \times b$. § 385

[The area of a parallelogram equals the product of its base and altitude.]

Therefore area triangle $ABC = \frac{1}{2} a \times b$. Ax. 8

Q. E. D.

391. COR. I. *Triangles having equal bases and equal altitudes are equivalent.*

392. COR. II. *Any two triangles are to each other as the products of their bases and altitudes.*

$$\left(\frac{P}{P'} = \frac{\frac{1}{2}ab}{\frac{1}{2}a'b'} = \frac{ab}{a'b'}.\right)$$

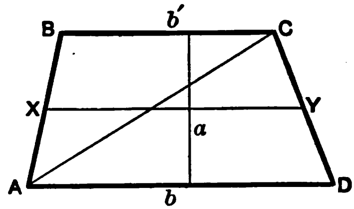
393. COR. III. *Two triangles having equal bases are to each other as their altitudes.*

394. COR. IV. *Two triangles having equal altitudes are to each other as their bases.*

395. *Def.*—The **altitude** of a trapezoid is the perpendicular distance between its bases.

PROPOSITION VII. THEOREM

396. *The area of a trapezoid equals the product of its altitude and one-half the sum of its bases.**



GIVEN—the trapezoid $ABCD$ with altitude a and bases b and b'.

TO PROVE the area of $ABCD = \frac{1}{2}(b+b')a$.

Draw the diagonal AC.

Then area triangle $ADC = \frac{1}{2}ab$, area triangle $ABC = \frac{1}{2}ab'$. § 390

[The area of a triangle equals one-half the product of its base and altitude.]

Adding, area trapezoid $ABCD = \frac{1}{2}ab + \frac{1}{2}ab'$. Ax. 11

$= \frac{1}{2}(b+b')\,a$. Q. E. D.

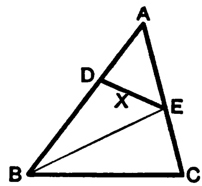
397. COR. *The area of a trapezoid equals the product of its altitude and the line joining the middle points of the non-parallel sides.*

Hint.—Combine § 135 with the above proposition.

* The ancient Egyptians attempted to find the area of a field in the form of a trapezoid, in which $AB = CD$, by multiplying half the sum of its parallel sides by *one of its other sides*, an incorrect method.

PROPOSITION VIII. THEOREM

398. *The areas of two triangles which have an angle of one equal to an angle of the other are to each other as the products of the sides including those angles.*



GIVEN—the triangles ADE and ABC placed so that their equal angles coincide at A.

TO PROVE $$\frac{\text{area } ADE}{\text{area } ABC}=\frac{AD\times AE}{AB\times AC}.$$

Draw BE and denote the triangle ABE by X.

Then, regarding the bases of X and ADE as AB and AD, they will have a common altitude, the perpendicular from E to AB. Likewise X and ABC have bases AE and AC and a common altitude, the perpendicular from B to AC.

Therefore $$\frac{\text{area } ADE}{\text{area } X}=\frac{AD}{AB}$$

and $$\frac{\text{area } X}{\text{area } ABC}=\frac{AE}{AC}.$$ § 39

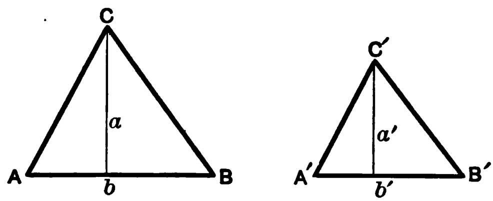
[Triangles having equal altitudes are to each other as their bas... s.]

Multiplying, $$\frac{\text{area } ADE}{\text{area } ABC}=\frac{AD\times AE}{AB\times AC}.$$ ...des at

Q. E. D.

PROPOSITION IX. THEOREM

399. *The areas of two similar triangles are to each other as the squares of any two homologous sides.*



GIVEN—two similar triangles ABC and $A'B'C'$, b and b' being homologous sides.

TO PROVE $$\frac{\text{area } ABC}{\text{area } A'B'C'} = \frac{b^2}{b'^2}.$$

Draw the altitudes a and a'.

Then $$\frac{\text{area } ABC}{\text{area } A'B'C'} = \frac{a \times b}{a' \times b'} = \frac{a}{a'} \times \frac{b}{b'}. \qquad \S\ 392$$

[Two triangles are to each other as the products of their bases and altitudes.]

But $$\frac{a}{a'} = \frac{b}{b'}. \qquad \S\ 290$$

[Homologous altitudes of similar triangles have the same ratio as homologous sides.]

ubstitute, in the previous equation, $\frac{b}{b'}$ for $\frac{a}{a'}$.

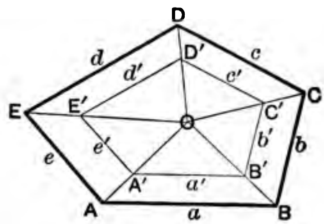
Then $$\frac{\text{area } ABC}{\text{area } A'B'C'} = \frac{b}{b'} \times \frac{b}{b'} = \frac{b^2}{b'^2}. \qquad \text{Q. E. D.}$$

SUMMARY: $$\frac{\text{area } ABC}{\text{area } A'B'C'} = \frac{a \times b}{a' \times b'} = \frac{a}{a'} \times \frac{b}{b'} = \frac{b}{b'} \times \frac{b}{b'} = \frac{b^2}{b'^2}.$$

400. *Exercise.*—Prove the last proposition by means of Proposition VIII.

PROPOSITION X. THEOREM

401. *The areas of two similar polygons are to each other as the squares of any two homologous sides.*



GIVEN—the similar polygons $ABCDE$ and $A'B'C'D'E'$, with sides a, b, c, d, e, and a', b', c', d', e', and areas M and M' respectively.

TO PROVE $$\frac{M}{M'}=\frac{a^2}{a'^2}.$$

If $ABCDE$ and $A'B'C'D'E'$ are radially placed so that O, the centre of similitude, is within the two polygons, the triangles OAB, OBC, OCD, etc., are respectively similar to $OA'B'$, $OB'C'$, $OC'D'$, etc. § 385

Then $\dfrac{\text{area } OAB}{\text{area } OA'B'}=\dfrac{a^2}{a'^2}$, $\dfrac{\text{area } OBC}{\text{area } OB'C'}=\dfrac{b^2}{b'^2}$, $\dfrac{\text{area } OCD}{\text{area } OC'D'}=\dfrac{c^2}{c'^2}$, etc. § 399

[The areas of two similar triangles are to each other as the squares of any two homologous sides.]

But $$\frac{a^2}{a'^2}=\frac{b^2}{b'^2}=\frac{c^2}{c'^2}=\text{etc.}$$ § 274

Hence $$\frac{\text{area } OAB}{\text{area } OA'B'}=\frac{\text{area } OBC}{\text{area } OB'C'}=\frac{\text{area } OCD}{\text{area } OC'D'}=\text{etc.}=\frac{a^2}{a'^2}.$$ Ax. 1

Therefore $\dfrac{\text{area } OAB + \text{area } OBC + \text{area } OCD + \text{etc.}}{\text{area } OA'B' + \text{area } OB'C' + \text{area } OC'D' + \text{etc.}} = \dfrac{a^2}{a'^2}$. § 265

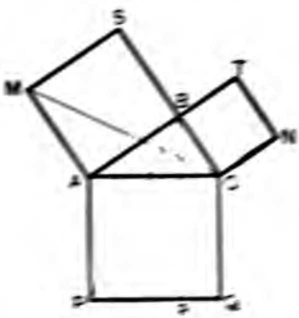
But area OAB + area OBC + area OCD + etc. $= M$, Ax. 11
and area $OA'B'$ + area $OB'C'$ + area $OC'D'$ + etc. $= M'$.

Therefore $\dfrac{M}{M'} = \dfrac{a^2}{a'^2}$. Q. E. D.

402. Cor. Since $\dfrac{a}{a'}$ = ratio of similitude, ***the ratio of the areas of two similar polygons equals the square of their ratio of similitude.***

PROPOSITION XL. THEOREM

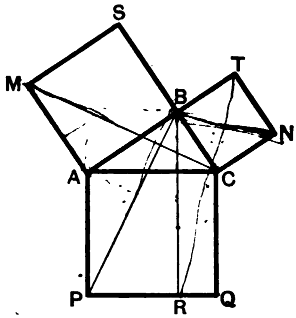
403. *The square described on the hypotenuse of a right triangle is equivalent to the sum of the squares on the other two sides.**



[illegible] the right triangle ABC and the squares described on its three [illegible]

[illegible]—area of square A[illegible] = area of square [illegible] + area of square [illegible]

* Proposition XL was discovered by [illegible] known as the Pythagorean theorem [illegible] about 500 B.C. [illegible]



Now ABT and CBS are straight lines. § 29

Join MC and BP and draw BR parallel to AP.

Triangle* AMC = triangle ABP. § 79

[Having two sides and the included angle equal, viz., $AM = AB$, being sides of a square; likewise $AC = AP$, and angle CAM = angle PAB, since each consists of a right angle and the common angle BAC.]

But rectangle $AR \Bumpeq$ twice triangle ABP. §§ 381, 390

[Having the same base AP and the same altitude, the distance between the parallels AP and BR.]

Likewise square $BM \Bumpeq$ twice triangle AMC.

Therefore rectangle $AR \Bumpeq$ square BM. Ax. 7

Likewise we may prove

rectangle $CR \Bumpeq$ square BN.

Adding,

rect. AR + rect. $CR \Bumpeq$ sq. BM + sq. BN.

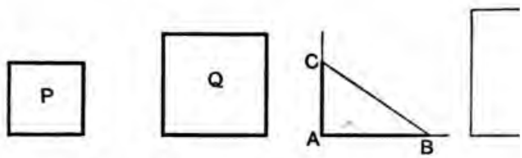
Or sq. $AQ \Bumpeq$ sq. BM + sq. BN. Q. E. D.

404. COR. *The square on either side about the right angle is equivalent to the difference of the squares on the hypotenuse and on the other side.*

* The e''e will interpret this equality by conceiving the triangle AMC to turr ivot until AM falls on AB.

405. *Remark.*—Proposition XV., Book III.,
the preceding proposition in that the squares
in the former referred to the *algebraic* squares,
second power of the numbers representing the s
as in the latter case the squares are *geometric.*]
the algebraic square measures the geometric sq
the truth of either of the two propositions involv
of the other.

406. CONSTRUCTION. *To construct a squar
to the sum of two given squares.*



GIVEN two squares P and Q.

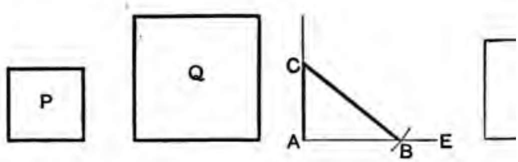
TO CONSTRUCT a square equivalent to $P+Q$.

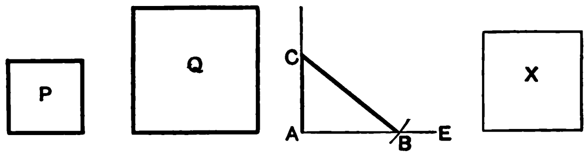
Construct a right angle A and on its sides lay
AC equal respectively to the sides of Q and P.

Construct the square X having its side equal t

X is the required square. (Why?)

407. CONSTRUCTION. *To construct a squar
to the difference of two given squares.*





GIVEN two squares, P and Q, of which P is the smaller.

TO CONSTRUCT a square equivalent to $Q - P$.

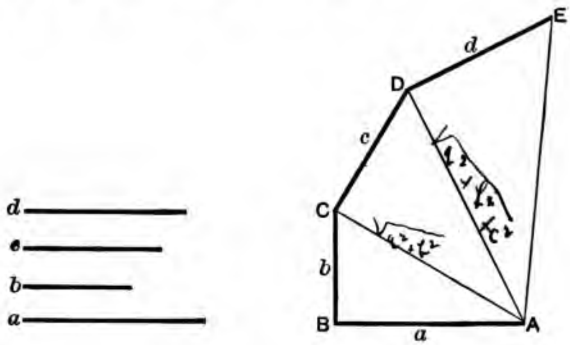
Construct a right angle A, and on one side lay off AC equal to the side of P.

Then from C as a centre, with the side of Q as a radius, describe an arc cutting AE at B.

Construct the square X having its side equal to AB.

X is the required square. (Why?) Q. E. F.

408. CONSTRUCTION. *To construct a square equivalent to the sum of any number of given squares.*



GIVEN a, b, c, d, the sides of given squares.

TO CONSTRUCT — a square equivalent to the sum of these given squares.

Draw AB equal to a.

At B draw BC perpendicular to AB and equal to b; join AC.

At C draw CD perpendicular to AC and equal to c; join AD.

At D draw DE perpendicular to AD and equal to d; join AE.

The square constructed on AE as a side is the square required.

Proof.—

Sq. on $AE \approx$ sq. on $d +$ sq. on AD.

$\approx$ sq. on $d +$ sq. on $c +$ sq. on AC.

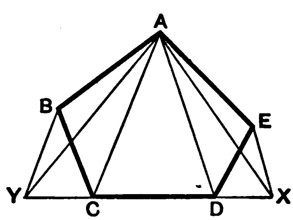
$\approx$ sq. on $d +$ sq. on $c +$ sq. on $b +$ sq. on a.

Q. E. F.

409. *Remark.*—The foregoing construction enables a draughtsman to construct a line whose length is equal to any square root.

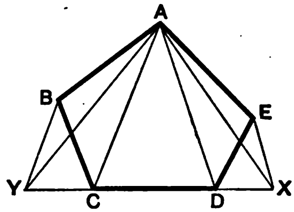
Thus suppose we wish to construct a line equal to $\sqrt{3}$ inches. Lay off a, b, c, one inch each; then $AD = \sqrt{3}$ inches.

410. CONSTRUCTION. *To construct a triangle equivalent to a given polygon.*



GIVEN the polygon $ABCDE$.

TO CONSTRUCT a triangle equivalent to it.



Join any two alternate vertices as A and D.

Draw EX parallel to AD and meeting CD produced at X. Join AX.

The polygon $ABCX$ has one less side than the original polygon, but is equivalent to it.

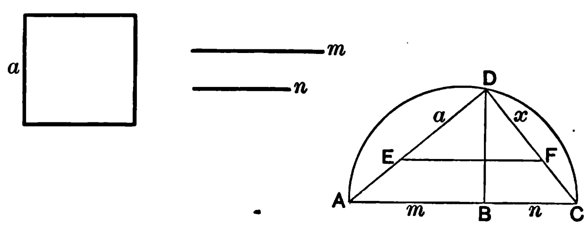
For the part $ABCD$ is common,

and triangle $ADE \Bumpeq$ triangle ADX. § 391

[Having the same base AD and the same altitude, the distance between the parallels AD and EX.]

In like manner reduce the number of sides of the new polygon $ABCX$, and thus continue until the required triangle AXY is obtained. Q. E. F.

411. CONSTRUCTION. *To construct a square which shall have a given ratio to a given square.*



GIVEN—a the side of a given square and $\frac{n}{m}$ the given ratio.

TO CONSTRUCT—a square which shall have the ratio $\frac{n}{m}$ to the given square.

Draw the straight line AB equal to m a
making BC equal to n.

Upon AC as a diameter construct a semicir

Erect the perpendicular BD meeting the ci
D, and join DA and DC.

On DA lay off DE equal to a and draw EF

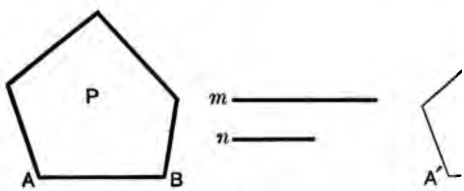
Then DF, or x, is the side of the square req

Proof: $$\frac{\text{square on } x}{\text{square on } a} = \frac{x^2}{a^2}$$

$$= \left(\frac{x}{a}\right)^2 = \left(\frac{DC}{DA}\right)^2 = \frac{\overline{DC}^2}{\overline{DA}^2} = \frac{BC}{AB}.$$

$$= \frac{n}{m}.$$

412. CONSTRUCTION. *To construct a polyg*
given polygon and having a given ratio to it.



GIVEN the polygon P, and the ratio $\frac{n}{m}$.

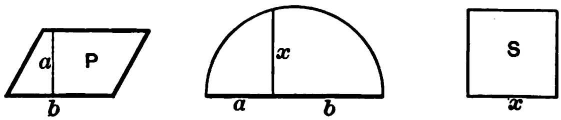
TO CONSTRUCT—a polygon similar to P, and whick
n is to m.

Find a line $A'B'$ such that the square upo
the square upon AB as n is to m.

Upon $A'B'$, as the homologous side to AE,
required similar polygon X.

Proof: $$\frac{X}{P} = \frac{\overline{A'B'}^2}{\overline{AB}^2} = \frac{n}{m}. \quad \text{(Why}$$

413. CONSTRUCTION. ***To construct a square equivalent to a given parallelogram.***



GIVEN a parallelogram P with base b and altitude a.

TO CONSTRUCT a square equivalent to P.

Construct x a mean proportional between a and b. § 316

Upon x construct the required square S.

Proof.—By construction $\frac{a}{x}=\frac{x}{b}$.

Hence $x^2=a\times b$. § 250

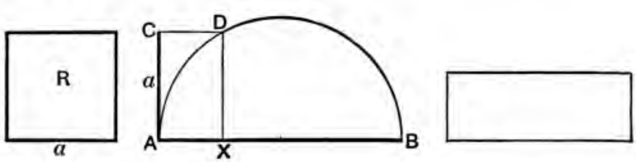
That is, area S=area P. §§ 383, 385

Q. E. D.

414. *Exercise.*—Show that a square can be constructed equivalent to a given triangle by taking for its side a mean proportional between the altitude and half the base.

415. *Exercise.*—Show that a square can be constructed equivalent to a given polygon by first reducing the polygon to an equivalent triangle and then constructing a square equivalent to the triangle.

416. CONSTRUCTION. *To construct a rectangle equivalent to a given square, and having the sum of its base and altitude equal to a given line.*



GIVEN—a, the side of the given square R, and AB, the given line.

TO CONSTRUCT—a rectangle equivalent to R and having its base and altitude together equal to AB.

Upon AB as a diameter construct a semicircle.

Draw CD parallel to AB and at a distance from it equal to a.

From D the intersection of CD with the circumference draw DX perpendicular to AB.

The rectangle having AX for its altitude and XB for its base is the required rectangle.

Proof: $\frac{AX}{DX}=\frac{DX}{XB}$. § 315

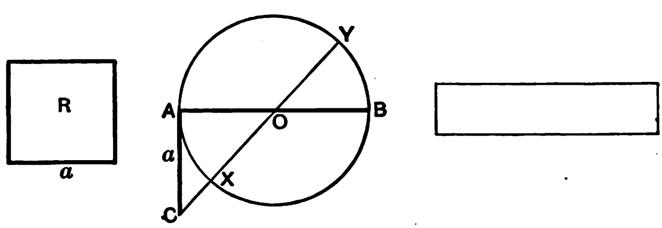
Hence $AX \times XB=\overline{DX}^2$. § 250

That is, area rectangle = area square. §§ 381, 383

Also $AX+XB=AB$. Q. E. F.

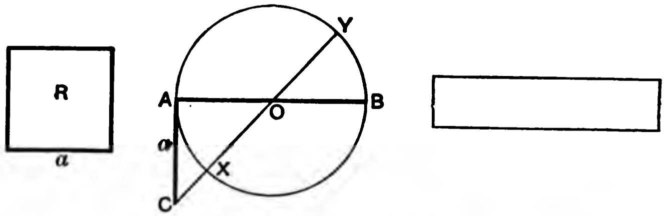
417. *Remark.*—§ 416 may be stated: To find two straight lines of which the sum and product are given.

418. CONSTRUCTION. *To construct a rectangle equivalent to a given square, and having the difference of its base and altitude equal to a given line.*



GIVEN a, the side of the square R, and the line AB.

TO CONSTRUCT—a rectangle equivalent to R, and having the difference of its base and altitude equal to AB.



Upon AB as a diameter construct a circumference.

At A draw the tangent AC equal to a.

Draw CXY through the centre meeting the circumference in X and Y.

Then the rectangle having its base equal to CY and its altitude equal to CX is the required rectangle.

Proof: $$\frac{CX}{a}=\frac{a}{CY}.$$ § 321

Whence $$CX \times CY = a^2.$$ § 250

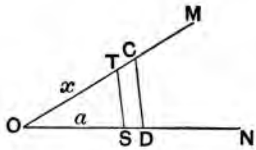
Or, area rectangle = area square. §§ 381, 383

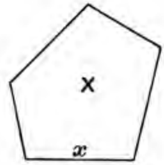
Also XY, the difference between CY and CX, is a diameter of the circle, and therefore equal to AB. Q. E. F.

419. *Remark.*—§ 418 may be stated: To find two straight lines of which the difference and product are given.

420. CONSTRUCTION. *To construct a polygon similar to a given polygon and equivalent to another given polygon.**

* Pythagoras (about 550 B.C.) first solved this problem.





GIVEN the polygons P and Q.

TO CONSTRUCT—a polygon similar to P and equivalent to Q.

Construct squares equivalent to P and Q. § 415

Let n and m be the sides of these squares.

From any point O draw two lines OM and ON, and on these lay off OC equal to m and OD equal to n. On OD lay off OS equal to a, a side of P.

Draw parallels giving the fourth proportional OT. § 282

Upon OT, or x, as a side homologous to a, construct a polygon X similar to P. It will also be equivalent to Q.

Proof: $$\frac{X}{P}=\frac{x^2}{a^2}=\frac{m^2}{n^2}=\frac{\text{sq. on } m}{\text{sq. on } n}=\frac{Q}{P}. \quad \text{(Why?)}$$

Therefore X is equivalent to Q and is similar to P by construction. Q. E. F.

PROBLEMS OF DEMONSTRATION

421. The square on the base of an isosceles triangle, whose vertical angle is a right angle, is equivalent to four times the triangle.

422. A quadrilateral is divided into two equivalent triangles by one of its diagonals, if the other diagonal is bisected by the first.

423. The four triangles formed by drawing the diagonals of a parallelogram are all equivalent.

424. If from the middle point of one of the diagonals of a quadrilateral straight lines are drawn to the opposite vertices, these two lines divide the figure into two equivalent parts.

425. If the sides of any quadrilateral are bisected and the points of bisection successively joined, the included figure will be a parallelogram equal in area to half the original figure.

426. A trapezoid is divided into two equivalent parts by the straight line joining the middle points of its parallel sides.

427. The triangle formed by joining the middle point of one of the non-parallel sides of a trapezoid to the extremities of the opposite side is equivalent to one-half the trapezoid.

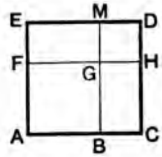
428. If the three sides of a right triangle are the homologous sides of similar polygons described upon them, then the polygon described upon the hypotenuse is equivalent to the sum of the polygons described upon the other two sides.

429. If M is the intersection of the medians of a triangle ABC, the triangle AMB is one-third of ABC.

430. If from the middle point of the base of a triangle lines parallel to the sides are drawn, the parallelogram thus formed is equivalent to one-half the triangle.

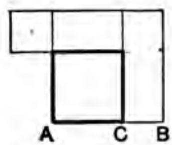
431. Any straight line drawn through the intersection of the diagonals of a parallelogram divides the parallelogram into two equivalent parts.

432. The square described upon the sum of two straight lines is equivalent to the sum of the squares described upon the two lines plus twice their rectangle.



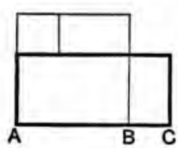
Hint.—Let AB and BC be the given lines.

433. The square described upon the difference of two straight lines is equivalent to the sum of the squares described upon the two lines minus twice their rectangle.



Hint.—Let AB and BC be the given lines.

434. The rectangle whose sides are the sum and the difference of two straight lines is equivalent to the difference of the squares described upon the two lines.



Hint.—Let AB and BC be the given lines.

Question.—To what three formulas of algebra* do the last three problems correspond?

* Euclid gave the geometric proofs of §§ 432–4; but though he may have translated them into algebra, he was probably not acquainted with the algebraic proof. To-day we find it easier to obtain the algebraic formulas first, and then give them the geometric interpretation. This is true in a multitude of cases where the opposite was true among the Greeks.

PROBLEMS OF CONSTRUCTION

435. To divide a triangle into three equivalent triangles by straight lines from one of the vertices to the side opposite.

436. To construct an isosceles triangle equivalent to any given triangle, and having the same base.

437. On a given side, to construct a triangle equivalent to any given triangle.

438. Having given an angle and one of the including sides, to construct a triangle equivalent to a given triangle.

439. To construct a right triangle equivalent to a given triangle.

440. To construct a right triangle equivalent to a given triangle, and having its base equal to a given line.

441. On a given hypotenuse to construct a right triangle equivalent to a given triangle. When is the problem impossible?

442. To draw a straight line through the vertex of a given triangle so as to divide it into two parts having the ratio 2 to 5.

443. To bisect a triangle by a straight line drawn from a given point in one of its sides. § 398

444. On a given side to construct a rectangle equivalent to a given square.

445. To construct a square equivalent to a given triangle.

446. To construct a square equivalent to the sum of two given triangles.

447. On a given side to construct a rectangle equivalent to the sum of two given squares.

448. To construct a square which shall have a given ratio to a given hexagon.

449. Through a given point within any parallelogram to draw a straight line dividing it into two equivalent parts.

PROBLEMS FOR COMPUTATION

450. (1.) Find the area of a parallelogram one of whose sides is 37.53 m., if the perpendicular distance between it and the opposite side is 2.95 dkm.

(2.) Required the area of a rhombus if its diagonals are in the ratio of 4 to 7, and their sum is 16.

(3.) In a right triangle the perpendicular from the vertex of the right angle to the hypotenuse divides the hypotenuse into the segments m and n. Find the area of the triangle.

(4.) If the hypotenuse of an isosceles right triangle is 30 ft., find the number of ares in its area.

(5.) Find the area of an isosceles right triangle if the hypotenuse is equal to a.

(6.) If one of the equal sides of an isosceles triangle is 17 dkm. in length and its base is 30 m., find the area of the triangle.

(7.) Find the area of an isosceles triangle if one of the equal sides is a and its base is b.

(8.) If in the above example $a = 17.163$ hm. and $b = 27.395$ hm., how many acres are there in the triangle?

(9.) Find the area of an equilateral triangle if one of the sides equals 16 m.

(10.) If the side of an equilateral triangle is a, find its area.

(11.) If each side of a triangular park measures 196.37 rds., how many hectares does it contain?

(12.) If the perimeter of an equilateral triangle is 523.65 ft., find its area.

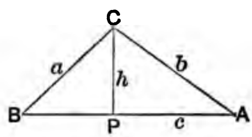
(13.) Find the area of a triangle, if two of its sides are 6 in. and 7 in. and the included angle is 30°.

(14.) Show that, if a and b are the sides of a triangle, the area is $\frac{1}{4}ab$, when the included angle is 30° or 150°; $\frac{1}{4}ab\sqrt{2}$, when the included angle is 45° or 135°; $\frac{1}{4}ab\sqrt{3}$, when the included angle is 60° or 120°.

(15.) Find the area of a triangle, if two of its sides are 43.746 mm. and 15.691 mm., and the included angle is 120°.

(16.) How many square feet are there in the entire surface of a house 50 ft. long, 40 ft. wide, 30 ft. high at the corners, and 40 ft. high at the ridge-pole?

(17.) Find the area of a triangle whose sides are a, b, and c.



Solution.—The area of the triangle $ABC = \frac{c}{2} \times h$.

But $$h = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}. \qquad \text{§ 373(31)}$$

Whence $$\text{area} = \frac{c}{2} \times \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}$$
$$= \sqrt{s(s-a)(s-b)(s-c)}.$$

(18.) Find the area of a triangle whose sides are 119.3 m., 147.35 m., and 7 dkm.

(19.) Required the area of the quadrilateral $ABCD$, if the four sides AB, BC, CD, and DA measure respectively 63.57, 113.29, 39.637, and 156 ft., and the diagonal $AC = 150.26$ ft.

(20.) If the bases of a trapezoid are respectively 97 m. and 133 m., and its area is 46 ares, find its altitude.

(21.) Find the area of a trapezoid of which the bases are 73 ft. and 57 ft., and each of the other sides is 17 ft.

(22.) Find the area of a trapezoid of which the bases are a and b and the other sides are each equal to d.

(23.) If in the triangle ABC a line MN is drawn parallel to the side AC so that the smaller triangle which it cuts off equals one-third of the whole triangle, find MN in terms of AC.

(24.) Through a triangular field a path runs from one corner to a point in the opposite side 204 yds. from one end, and 357 yds. from the other. What is the ratio of the two parts into which the field is divided?

(25.) If a square and a rhombus have equal perimeters, and the altitude of the rhombus is four-fifths its side, compare the areas of the two figures.

(26.) The altitude upon the hypotenuse of an isosceles right triangle is 3.1572 m. Find the side of an equivalent square.

(27.) If the areas of two triangles of equal altitude are 9 hectares and 324 ares respectively, what is the ratio of their bases?

(28.) A triangle and a rectangle are equivalent. (*a.*) If their bases are equal find the ratio of their altitudes. (*b.*) Compare their bases if their altitudes are equal.

(29.) Two homologous sides of two similar polygons are respectively 12 m. and 36 m. in length, and the area of the first is 180 sq. m. What is the area of the second?

(30.) Two similar fields together contain 579 hectares. What is the area of each if their homologous sides are in the ratio of 7 to 12?

(31.) In a triangle having its base equal to 24 in. and an area of 216 sq. in., a line is drawn parallel to the base through a point 6 in. from the opposite vertex. Find the area of the smaller triangle thus formed.

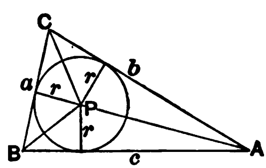
(32.) The altitude of a triangle is a and its base is b; the altitude, homologous to a, of another triangle, similar to the first, is c. Find the altitude, base, and area of a triangle similar to the given triangles and equivalent to their sum.

(33.) Construct a square equivalent to the sum of the squares whose sides are 20, 16, 9, and 5 cm.

(34.) If the sides of a triangle are 113.61 cm., 97.329 cm., and 82.52 cm., find the areas of the parts into which it is divided by the bisector of the angle opposite the first side.

(35.) If to the base b of a triangle the line d is added, how much must be taken from its altitude h that its area may remain unchanged?

(36.) If the sides of a triangle are a, b, and c, find the radius of the inscribed circle.



Solution.—The area of the triangle $CBP=\frac{a}{2}\times r.$

The area of the triangle $CAP=\frac{b}{2}\times r.$

The area of the triangle $BAP=\frac{c}{2}\times r.$

The sum of these areas, or the area of the triangle ABC,

$$=\frac{a+b+c}{2}\times r=sr.$$

But by (17) the area of

$$ABC=\sqrt{s(s-a)(s-b)(s-c)}.$$

Therefore
$$sr=\sqrt{s(s-a)(s-b)(s-c)}$$
$$r=\frac{1}{s}\sqrt{s(s-a)(s-b)(s-c)}$$
$$=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

(37.) If the sides of a triangle are 173.52 cm., 125.3 cm., and 96.357 cm., find the radius of the inscribed circle.

PLANE GEOMETRY

BOOK V

REGULAR POLYGONS AND CIRCLES. SYMMETRY WITH RESPECT TO A POINT

451. *Defs.*—A figure turns **half-way round** a point, if a straight line of the figure passing through the point turns through 180°, i. e., half of 360°.

A figure turns **one-third-way round** a point, if a straight line of the figure passing through the point turns through 120°, i. e., one-third of 360°.

In general, a figure turns **one-nth way round** a point if a straight line of the figure passing through the point turns through one-n^{th} of 360°.

452. *Exercise.*—If a figure is turned half-way round on a point as a pivot, i. e., so that *one* straight line of the figure passing through that point turns through 180°, prove that *every other* straight line of the figure passing through that point turns through 180°.

453. *Exercise.*—In the same case, prove that every straight line not passing through the pivot makes after the rotation an angle of 180° with its original position.

454. *Exercise.*—If a figure turns one-third way round, prove that every straight line, whether passing through the pivot or not, makes after the rotation an angle of 120° with its original position.

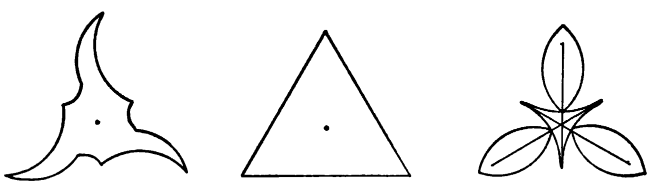
455. *Exercise.*—If a figure turns one-n^{th} way round, prove that every straight line of the figure makes after the rotation an angle equal to $\frac{1}{n}$ of 360° with its original position.

456. *Remark.*—Hence we see the propriety of saying that when one straight line of the figure turns through an angle, the whole figure turns through the same angle.

457. *Defs.*—A figure was defined to be symmetrical with respect to a point, called the **centre of symmetry** (§ 40), if, on being turned *half-way round* on that point as a pivot, the figure coincides with its original position or impression.

To distinguish this kind of symmetry from those which follow, it may be called **two-fold symmetry** with respect to a point.

458. *Def.*—A figure has **three-fold** symmetry with respect to a point, if, on being turned *one-third* way round on that point as a pivot, it coincides with its original impression.

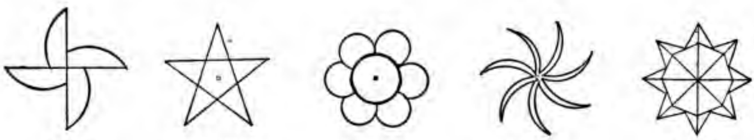


FIGURES POSSESSING THREE-FOLD SYMMETRY WITH RESPECT TO A POINT

A figure which coincides with its original when turned one-third way round must also coincide when turned *two*-thirds. For, since it *coincides* after the first third, it may then be regarded as the original figure, and will therefore coincide when turned one-third again. When turned the third third the figure has completed one revolution, and each part is in its original position. It is easy to copy one of the above figures on tracing-paper or card-board, cut it out, fit it again to the page, stick a pin through its centre, and turn the figure one-third way round. In Propositions I. and II. it is convenient to think of the original diagram as fixed on the page, while another diagram, as the card-board, revolves upon it.

459. *Defs.*—We may define likewise four-fold, five-fold, etc., symmetry. In general a figure has **n-fold symmetry** with respect to a point, called the **centre of symmetry,** if, on being turned about that point one-n^{th} of a revolution, it coincides with its original impression.

Such a figure will also coincide if turned an n^{th} of a revolution a second, third, fourth time, etc. For after the first n^{th} it becomes the *original figure*, and will therefore coincide when turned one-n^{th} again.

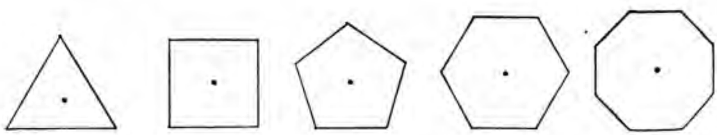


4-FOLD SYMMETRY 5-FOLD SYMMETRY * 6-FOLD SYMMETRY 7-FOLD SYMMETRY 8-FOLD SYMMETRY

460. *Defs.*—A triangle is **regular,** if it has three-fold symmetry with respect to a point. The point is called the **centre of the triangle.**

A quadrilateral is **regular,** if it has four-fold symmetry; a pentagon if it has five-fold symmetry, etc.

In general a polygon of n sides is **regular,** if it has n-fold symmetry. The centre of symmetry is called the **centre of the polygon.**



REGULAR TRIANGLE REGULAR QUADRILATERAL REGULAR PENTAGON REGULAR HEXAGON REGULAR OCTAGON

* This figure was used as a badge by the secret society founded by Pythagoras about 550 B.C. for the pursuit of Mathematics and Philosophy. It was supposed to possess mysterious properties, and was called "Health."

PROPOSITION I. THEOREM

461. *Given a regular polygon:*

I. *All its sides are equal.*

II. *All its angles are equal.*

III. *A circle may be circumscribed about it, its centre being the centre of the polygon.*

IV. *A circle may be inscribed in it, its centre being the centre of the polygon.*

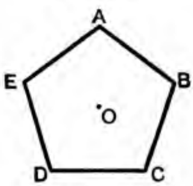


FIG. 1

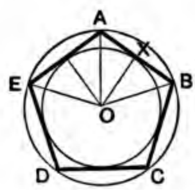


FIG. 2

GIVEN—$ABCDE$, a regular polygon of n sides with centre O.

TO PROVE—I. Its sides are equal.

II. Its angles are equal.

III. A circle can be circumscribed, with centre O.

IV. A circle can be inscribed, with centre O.

I. (Fig. 1.) By definition, the polygon will, after being turned about O one-n^{th} of a revolution, coincide with its original impression. § 460

Any side as AB must therefore take the position previously occupied by some other side.

Since each turn is one-n^{th} of a revolution, n turns are necessary before AB resumes its original position.

Hence in a complete revolution AB must coincide in succession with the n different sides of the polygon.

Hence AB is equal to each of the other sides, and they are all equal to each other. Q. E. D.

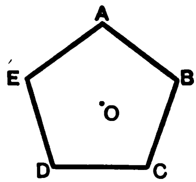


FIG. 1

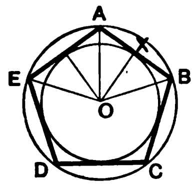


FIG. 2

II. (Fig. 1.) Likewise any angle, as A, must in the n turns necessary for a complete revolution coincide in succession with the n different angles of the polygon.

Hence the angles are all equal. Q. E. D.

III. (Fig. 2.) Since the vertex A always remains at the same distance from O, it describes a circumference whose centre is O.

But it has been shown that the point A coincides successively with B, C, D, etc.

Hence the circumference described by A passes through B, C, D, etc.

That is, this circumference is circumscribed about the polygon and has for its centre the point O. § 218

Q. E. D.

IV. (Fig. 2.) Consider a perpendicular from O upon any side, as OX upon AB.

As the figure revolves, AB coincides successively with each of the other sides, and therefore OX becomes successively perpendicular to each side.

Hence the circumference generated by X, whose radius is OX, passes through the feet of all the perpendiculars from O to the sides.

The sides are therefore all tangent to this circle. § 173

That is, the circle is inscribed in the polygon, and has its centre at O. § 214

Q. E. D.

462. COR. I. *A regular triangle is an equilateral and equiangular triangle. A regular quadrilateral is a square.*

463. COR. II. *Each angle of a regular polygon is* $\frac{2n-4}{n}$ *right angles* (*n being the number of sides*).

Hint.—By § 66 the sum of all the angles is $2n-4$ right angles.

464. *Def.*—The **radius** of a regular polygon is the radius of the circumscribed circle, that is, the line from the centre to a vertex.

465. *Def.*—The **apothem** of a regular polygon is the radius of the inscribed circle, that is, the perpendicular from the centre to a side.

466. COR. III. *The angles at the centre of a regular polygon between successive radii are all equal, and each is one-n^{th} of four right angles.*

467. *Def.*—Any one of these angles is usually spoken of simply as the **angle at the centre.**

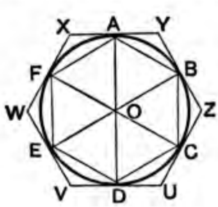
468. COR. IV. *The angle at the centre of a regular polygon is bisected by the apothem.*

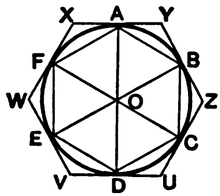
PROPOSITION II. THEOREM

469. *If the circumference of a circle be subdivided into three or more equal arcs:*

I. *Their chords form a regular inscribed polygon, whose centre is the centre of the circle.*

II. *The tangents at the points of division form a regular circumscribed polygon, whose centre is the centre of the circle.*





GIVEN—a circle whose centre is O and whose circumference is divided into n equal arcs at the points A, B, C, D, etc.

TO PROVE—I. The n chords AB, BC, etc., form a regular polygon, with centre O.

II. The n tangents XAY, YBZ, etc., form a regular polygon, with centre O.

I. Revolve the figure one-n^{th} of 360°.

As the figure is turned, the circumference slides along itself. § 159

Since the arcs are each equal to one-n^{th} of the circumference, when A reaches B, B will reach C, C will reach D, etc.

That is, each vertex of the revolved polygon coincides with a vertex of the original polygon.

Since the vertices coincide, the sides which connect them must also coincide. Ax. *a*

Hence the whole polygon coincides with its original impression, and is therefore regular. § 460

Q. E. D.

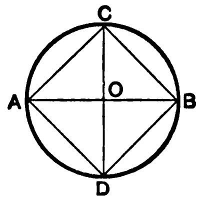
II. We have just proved that when the figure is revolved one-n^{th}, the vertices A, B, C, etc., will coincide respectively with B, C, D, etc., and we know that the circumference will coincide with itself. § 159

Hence the tangents at A, B, C, etc., will coincide respectively with the tangents at B, C, D, etc. §§ 173, 18

Hence the whole circumscribed polygon will coincide with its original impression, and is therefore regular. § 460

Q. E. D.

470. CONSTRUCTION. *To inscribe a regular quadrilateral, or square, in a given circle.*



GIVEN a circle with centre O.

TO CONSTRUCT an inscribed square.

Draw two perpendicular diameters AB and CD.

Join their extremities.

$ACBD$ is the required square.

Proof.—The arcs AC, CB, BD, DA are equal. § 162

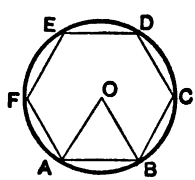
[Subtending equal angles at the centre.]

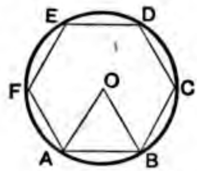
Hence $ACBD$ is a regular quadrilateral. § 469 I

Q. E. D.

471. *Remark.*—A regular polygon of eight sides can be inscribed by bisecting the arcs AC, CB, etc.; and, by continuing the process, regular polygons of sixteen, thirty-two, sixty-four, one hundred and twenty-eight, etc., sides can be inscribed.

472. CONSTRUCTION. *To inscribe a regular hexagon in a given circle.*





GIVEN a circle with centre O.

TO CONSTRUCT a regular inscribed hexagon.

Draw any radius OA.

With A as a centre and a radius equal to OA describe an arc intersecting the circumference at B.

AB is a side of the required regular inscribed hexagon.

Proof.—Join OB.

The triangle OAB is equilateral. Cons.

Hence angle O is 60°, i. e., one-sixth of 360°. § 74

Hence arc AB is one-sixth of the circumference. § 191

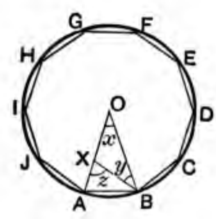
Therefore chord AB is a side of a regular inscribed hexagon. § 469 I

Q. E. D.

473. *Exercise.*—Show that a regular inscribed triangle is formed by joining the alternate vertices A, C, and E.

474. *Remark.*—A regular inscribed polygon of twelve sides can be formed by bisecting the arcs AB, BC, etc.; and, by continuing the process, regular polygons of twenty-four, forty-eight, ninety-six, etc., sides can be inscribed.

475. CONSTRUCTION. *To inscribe a regular decagon in a given circle.*



GIVEN a circle with centre O.

TO CONSTRUCT a regular inscribed decagon.

Divide a radius OA internally in extreme and mean ratio,

i. e., so that $$\frac{OA}{OX}=\frac{OX}{XA}.$$ § 335

With A as a centre and OX as a radius, describe an arc cutting the circumference at B.

AB is a side of the required regular inscribed decagon.

Proof.—Join BX and BO.

Substituting AB for its equal OX we have

$$\frac{OA}{AB}=\frac{AB}{AX}.$$

Hence triangles AOB and ABX are similar. § 285

[Having the angle A common and the including sides proportional.]

But AOB is isosceles. § 150

Therefore ABX is isosceles, and $AB=BX=OX$. Cons.

Whence OXB is isosceles, and angle y=angle x. § 71

Then angle $z=x+y=2x$. § 59

And angle $OBA=A=z=2x$. § 71

Hence, in the triangle AOB,

angle $OAB+OBA+x=5x=2$ right angles. § 58

Therefore $x=\frac{1}{5}$ of 2 right angles, or $\frac{1}{10}$ of 4 right angles.

And arc $AB=\frac{1}{10}$ of the circumference. § 191

Therefore chord AB=side of regular inscribed decagon.

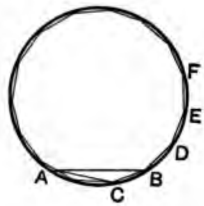
§ 469 I

Q. E. D.

476. *Exercise.*—Show that a regular pentagon is inscribed by joining the alternate vertices, A, C, E, G, I.

477. *Remark.*—A regular polygon of twenty sides is inscribed by bisecting the arcs AB, BC, etc., and, by continuing the process regular polygons of forty, eighty, etc., sides can be inscribed.

478. CONSTRUCTION. *To inscribe a regular pentedecagon in a given circle.*



GIVEN a circle AF.

TO CONSTRUCT—a regular inscribed pentedecagon.

Draw chord AB, the side of a regular inscribed hexagon. § 472

Draw chord AC, the side of a regular inscribed decagon. § 475

Then chord BC is a side of the required regular inscribed pentedecagon.

Proof: Arc AB is $\frac{1}{6}$ of the circumference.

Arc AC is $\frac{1}{10}$ of the circumference.

Hence Arc BC is $\frac{1}{6}-\frac{1}{10}$, or $\frac{1}{15}$ of the circumference.

Hence chord BC is the side of a regular inscribed polygon of fifteen sides. § 469 I

Q. E. D.

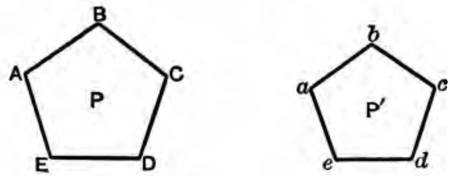
479. *Remark.*—A regular polygon of thirty sides can be inscribed by bisecting the arcs CB, BD, etc.; and, by continuing the process, regular polygons of sixty, one hundred and twenty, etc., sides can be inscribed.*

* We have seen how to inscribe polygons of

3, 6, 12, 24, 48, 96, etc., sides,
4, 8, 16, 32, 64, 128, etc., sides,
5, 10, 20, 40, 80, 160, etc., sides,
15, 30, 60, 120, 240, 480, etc., sides.

PROPOSITION III. THEOREM

480. *Two regular polygons of the same num*
are similar.



GIVEN—P and P', two regular polygons, each having n

TO PROVE P and P' are similar.

$$\left.\begin{array}{l} AB=BC=CD=\text{etc.} \\ ab=bc=cd=\text{etc.} \end{array}\right\}$$

Dividing, $$\frac{AB}{ab}=\frac{BC}{bc}=\frac{CD}{cd}=\text{etc.}$$

That is, the two polygons have their homol
proportional.

Also, since there are n angles in each polygon, e
either polygon contains $\frac{2n-4}{n}$ right angles.

That is, the two polygons are mutually equian
Therefore they are similar.

Up to the year 1796 these were the only regular polygons for whi
were known. In that year Gauss, the greatest mathematician o
century, then nineteen years of age, discovered a method of constr
of ruler and compasses, a regular polygon of 17 sides, and in gene
of $2^m(2^n+1)$ sides, m and n being integers, and (2^n+1) a
This method was given in the *Disquisitiones Arithmeticæ*, pub
In connection with this method Gauss enunciated the celebrat
only a limited class of regular polygons are constructible by rul

PROPOSITION IV. THEOREM

481. *In two regular polygons of the same number of sides, two corresponding sides are to each other as the radii or as the apothems.*

GIVEN—AB and $A'B'$, sides of regular polygons, each having the same number (n) of sides; and OA, $O'A'$, and OF, $O'F'$, the radii and apothems respectively.

TO PROVE $$\frac{AB}{A'B'}=\frac{OA}{O'A'}=\frac{OF}{O'F'}.$$

In the triangles OAB and $O'A'B'$,

angle O = angle O'. § 466

[Each being one-n^{th} of four right angles.]

Also $OA = OB$ § 150

and $O'A' = O'B'$.

Whence $$\frac{OA}{O'A'}=\frac{OB}{O'B'}.$$

Therefore the triangles are similar. § 285

Hence $$\frac{AB}{A'B'}=\frac{OA}{O'A'}.$$ § 274

And $$\frac{AB}{A'B'}=\frac{OF}{O'F'}.$$ § 290

Q. E. D.

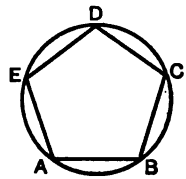
482. COR. I. *The perimeters of two regular polygons of the same number of sides are to each other as their radii or as their apothems.*

Hint.—Apply § 308.

483. COR. II. *The areas of two regular polygons of the same number of sides are to each other as the squares of their radii or as the squares of their apothems.*

PROPOSITION V. THEOREM

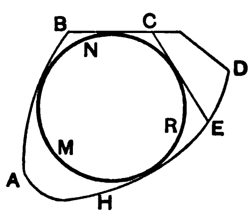
484. *The circumference of a circle is greater than the perimeter of an inscribed polygon.*



The proof is left to the student.

PROPOSITION VI. THEOREM

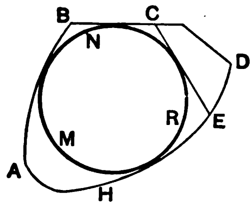
485. *The circumference of a circle is less than the perimeter of a circumscribed polygon or any enveloping line.*



GIVEN the circumference MNR.

TO PROVE—it is less than $ABCDEH$, any enveloping line.

Of all the lines enclosing the area MNR (of which the circumference MNR is one) there must be at least one *shortest* or *minimum* line.



The enveloping line $ABCDEH$ is not a minimum line, since we can obtain a shorter one by drawing a tangent CE.

For $CE < CDE$. § 7

Therefore $ABCEH < ABCDEH$. Ax. 4

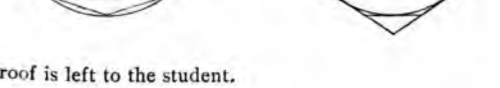
Likewise we may prove that *every* line enclosing MNR *except* the circumference is not minimum.

There remains therefore the circumference as the only minimum line. Q. E. D.

PROPOSITION VII. THEOREM

486. I. *If one regular inscribed polygon has twice as many sides as another, its perimeter and area are greater than those of the other.*

II. *If one regular circumscribed polygon has twice as many sides as another, its perimeter and area are less than those of the other.*



The proof is left to the student.

487. THEOREM. *If a variable x can be made less than any assigned quantity, the product of that variable and a decreasing quantity h can be made less than any assigned quantity.*

Let k be a constant greater than any value of h.

It has been proved that kx can be made less than any assigned quantity. § 187

But hx is always less than kx. Ax. 7

Hence hx can be made less than any assigned quantity.

488. COR. *If a variable x can be made less than any assigned quantity, then x^2 can be made less than any assigned quantity.*

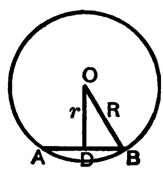
Hint.—Put x for h in the last theorem.

PROPOSITION VIII. LEMMA

489. *By doubling indefinitely the number of sides of a regular polygon inscribed in a given circle:*

I. *The apothem can be made to differ from the radius by less than any assigned quantity.*

II. *The square of the apothem can be made to differ from the square of the radius by less than any assigned quantity.*

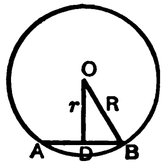


GIVEN—AB a side and r the apothem of a regular polygon inscribed in a circle whose radius is R.

TO PROVE—I. $R-r$ can be made as small as we please.

II. R^2-r^2 can be made as small as we please.

I. By doubling indefinitely the number of divisions of the circumference, the arc AB can be made as small as we please.



Therefore the chord AB, which is always less than the arc, can be made as small as we please.

Therefore DB, half of that chord, can be made as small as we please.

But $$R-r<DB.$$ § 137

Therefore $R-r$, which is always less than DB, can be made as small as we please. Q. E. D.

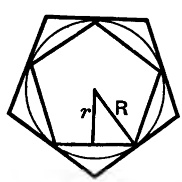
II. Since we can make DB as small as we please, we can also make $\overline{DB}^2$ as small as we please. § 488

But $$R^2-r^2=\overline{DB}^2.$$ § 318

Therefore we can make R^2-r^2, the equal of $\overline{DB}^2$, as small as we please. Q. E. D.

PROPOSITION IX. THEOREM

490. *The circumference of a circle is the limit which the perimeters of regular inscribed and circumscribed polygons approach when the number of their sides is doubled indefinitely; and the area of the circle is the limit of the areas of these polygons.*



GIVEN—P and p the perimeters, R and r the apothems, S and s the areas, respectively, of regular circumscribed and inscribed polygons of the same number of sides.

TO PROVE—I. The circumference of the circle is the common limit of P and p, when the number of sides is doubled indefinitely.

II. The area of the circle is the common limit of S and s, when the number of sides is doubled indefinitely.

I. Since the two regular polygons have the same number of sides, $$\frac{P}{p}=\frac{R}{r}.$$ § 482

By division $$\frac{P-p}{P}=\frac{R-r}{R}.$$ § 259

Or $$P-p=P\frac{R-r}{R}.$$

But, by doubling indefinitely the number of sides, $R-r$ can be made as small as we please. § 489 I

Hence $\frac{R-r}{R}$, the preceding variable divided by R, a constant quantity, can be made as small as we please. § 188

Hence $P\frac{R-r}{R}$, the preceding multiplied by P, a *decreasing* quantity (§ 486 II.), can be made as small as we please. § 487

Hence its equal $P-p$ can be made as small as we please.

But the circumference is always intermediate between P and p. §§ 484, 485

Therefore P and p, which can be made to differ *from each other* by less than any assigned quantity, can each be made to differ from the *intermediate quantity*, the circumference, by less than any assigned quantity.

But P and p can never equal the circumference. §§ 484, 485

Therefore by the definition of a limit the circumference is the common limit of P and p. § 185

Q. E. D.

II. Also, since the polygons are similar, § 480

$$\frac{S}{s}=\frac{R^2}{r^2}. \qquad \text{§ 483}$$

By division $$\frac{S-s}{S}=\frac{R^2-r^2}{R^2}.$$

Or $$S-s=S\frac{R^2-r^2}{R^2}.$$

But R^2-r^2 can be made as small as we please. § 489 II

Hence $\dfrac{R^2-r^2}{R^2}$, the preceding variable divided by R^2, a constant quantity, can be made as small as we please. § 188

Hence $S\dfrac{R^2-r^2}{R^2}$, the preceding multiplied by S, a *decreasing* quantity (§ 486 II.), can be made as small as we please. § 487

Hence its equal $S-s$ can be made as small as we please.

But the area of the circle is always intermediate between S and s. Ax. 10

Therefore S and s, which can be made to differ *from each other* by less than any assigned quantity, can each be made to differ from the *intermediate quantity*, the area of the circle, by less than any assigned quantity.

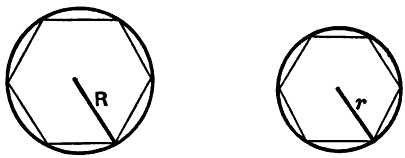
But S and s can never equal the area of the circle. Ax. 10

Therefore by the definition of a limit the area of the circle is the common limit of S and s. § 185

Q. E. D.

PROPOSITION X. THEOREM

491. *The ratio of the circumference of a circle to its diameter is the same for all circles.*



GIVEN—any two circles with radii R and r, and circumferences C and c respectively.

TO PROVE $\frac{C}{2R}=\frac{c}{2r}$.

Inscribe in the two circles regular polygons of the same number of sides, and call their perimeters P and p.

Then $\frac{P}{p}=\frac{R}{r}=\frac{2R}{2r}$. § 482

Hence $\frac{P}{2R}=\frac{p}{2r}$. § 256

As the number of sides of the two inscribed polygons is indefinitely doubled, P approaches C as its limit and p approaches c as its limit. § 490

Hence $\frac{P}{2R}$ approaches $\frac{C}{2R}$ as its limit,

and $\frac{p}{2r}$ approaches $\frac{c}{2r}$ as its limit. § 190

But always $\frac{P}{2R}=\frac{p}{2r}$.

Hence $\frac{C}{2R}=\frac{c}{2r}$. § 186

Q. E. D.

492. *Def.*—This uniform ratio of a circumference to its diameter is called π. It will be shown in § 502 that its value is approximately $3\frac{1}{7}$.

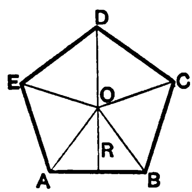
493. Cor. *The circumference of a circle is equal to its radius multiplied by* 2π.

Hint.—By definition $\frac{C}{2R} = \pi$.

494. *Exercise.*—The radius of a locomotive driving-wheel is 6 feet; how far does it roll on the track in one revolution?

PROPOSITION XI. THEOREM

495. *The area of a regular polygon is equal to half the product of its apothem and perimeter.*



Given—a regular polygon $ABCDE$, R its apothem, and P its perimeter.

To prove area polygon $= \frac{1}{2} R \times P$.

Draw from O the centre OA, OB, OC, etc.

The polygon is thus divided into as many triangles as it has sides.

The apothem R is their common altitude, and their bases are the sides of the polygon.

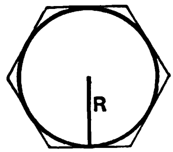
The area of *each* is $\frac{1}{2} R$ times its base. § 390

The area of *all* is $\frac{1}{2} R$ times the sum of their bases.

Or area polygon $= \frac{1}{2} R \times P$. Q. E. D.

PROPOSITION XII. THEOREM

496. *The area of a circle equals half the prod. dius and circumference.*



GIVEN—a circle with radius R, circumference C, and ar

TO PROVE $S=\frac{1}{2}R\times C$.

Circumscribe a regular polygon and call its p and area S'.

Then $S'=\frac{1}{2}R\times C'$.

[The area of a regular polygon equals half the product and perimeter.]

Let the number of sides of the regular circums gon be indefinitely increased.

C', the perimeter of the polygon, approache cumference, as its limit.

Hence $\frac{1}{2}R\times C'$ approaches $\frac{1}{2}R\times C$ as its limi

Also S' approaches S as its limit.

But *always* $S'=\frac{1}{2}R\times C'$.

Therefore $S=\frac{1}{2}R\times C$.

497. COR. I. *The area of a circle is* πR^2.

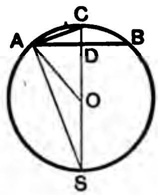
498. COR. II. *The area of a sector whose a*

$\frac{n}{360}(\pi R^2)$.

499. COR. III. *The areas of two circles are to the squares of their radii, or as the squares of the*

PROPOSITION XIII. PROBLEM

500. ***Given a circle of unit diameter and the side of a regular inscribed polygon, to find the side of a regular inscribed polygon of double the number of sides.***



GIVEN—the circle O of unit diameter, and AB, or s, the side of a regular inscribed polygon.

TO FIND—the length of AC, or x, a side of a regular polygon of double the number of sides.

Draw CS, the diameter perpendicular to AB.

Join AO and AS.

Now CAS is a right angle. § 202

And $AD=\frac{s}{2}$. § 167

Also $CS=1$, $AO=\frac{1}{2}$, $CO=\frac{1}{2}$. Cons.

Hence $\overline{AC}^2=CS\times CD$ § 312

$$=1\times CD=CD=CO-DO=\frac{1}{2}-DO$$

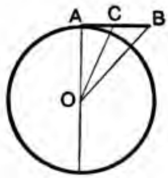
$$=\frac{1}{2}-\sqrt{\overline{AO}^2-\overline{AD}^2} \qquad \text{§ 318}$$

$$=\frac{1}{2}-\sqrt{\left(\frac{1}{2}\right)^2-\left(\frac{s}{2}\right)^2}=\frac{1-\sqrt{1-s^2}}{2}.$$

Therefore $$\boldsymbol{AC=x=\sqrt{\frac{1-\sqrt{1-s^2}}{2}}}.$$

PROPOSITION XIV. PROBLEM

501. *Given a circle of unit diameter and the side of a regular circumscribed polygon, to find the side of a regular circumscribed polygon of double the number of sides.*



GIVEN—the circle O of unit diameter and AB, or $\frac{s}{2}$, *half* the side of a regular circumscribed polygon.

TO FIND—AC, or $\frac{x}{2}$, *half* the side of a regular circumscribed polygon of double the number of sides.

Join OA, OC, OB.

Angle AOB is *half* the angle between successive radii of the first polygon. § 468

Angle AOC is *half* the angle between successive radii of the second polygon. § 468

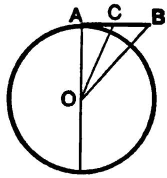
But the angle between successive radii in the second polygon is half that in the first. § 466

Therefore angle $AOC=\frac{1}{2}$ angle AOB, that is, OC bisects the angle AOB.

Hence $$\frac{AC}{CB}=\frac{AO}{OB},$$ § 327

or $$\frac{AC}{AB-AC}=\frac{AO}{\sqrt{\overline{AO}^2+\overline{AB}^2}}.$$

8



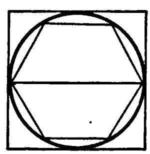
Substituting, $$\frac{\frac{x}{2}}{\frac{s}{2}-\frac{x}{2}}=\frac{\frac{1}{2}}{\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{s}{2}\right)^2}}.$$

Simplifying, $$\frac{x}{s-x}=\frac{1}{\sqrt{1+s^2}}.$$

Solving, $$x=\frac{s}{1+\sqrt{1+s^2}}.$$

PROPOSITION XV. PROBLEM

502. *To compute the ratio of the circumference of a circle to its diameter approximately.*



GIVEN a circle.

TO FIND—the ratio of its circumference to its diameter approximately, or the value of π.

Since the ratio π is the same for all circles (§ 491), it is sufficient to compute it for any one.

We select a circle of which the diameter is unity.

The radius of this circle will be $\frac{1}{2}$ and the side of a regular inscribed hexagon will be $\frac{1}{2}$; and of a circumscribed square 1.

Using the formula $x=\sqrt{\frac{1-\sqrt{1-s^2}}{2}}$ (§ 500), we form the following table giving the length of the sides of regular inscribed polygons of 6, 12, 24, etc., sides. The length of the perimeter is obtained by multiplying the length of one side by the number of sides.

INSCRIBED REGULAR POLYGONS

NO. SIDES	LENGTH OF SIDE	LENGTH OF PERIMETER
6	0.500000	3.000000
12	0.258819	3.105829
24	0.130526	3.132629
48	0.065403	3.139350
96	0.032719	3.141032
192	0.016362	3.141453
384	0.008181	3.141558

Using the formula $x=\frac{s}{1+\sqrt{1+s^2}}$ (§ 501), we form the following table giving the length of the sides and perimeters of regular circumscribed polygons of 4, 8, 16, etc., sides.

CIRCUMSCRIBED REGULAR POLYGONS

NO. SIDES	LENGTH OF SIDE	LENGTH OF PERIMETER
4	1.000000	4.000000
8	0.414214	3.313709
16	0.198912	3.182598
32	0.098492	3.151725
64	0.049127	3.144118
128	0.024549	3.142224
256	0.012272	3.141750
512	0.006136	3.141632

But the length of the circumference must be intermediate between the lengths of the circumscribed and inscribed poly-

gons. Hence it must be intermediate between 3.141558 and 3.141632. Hence 3.1416 is the nearest approximation to four decimal places.

Since the diameter of the circle is 1, the ratio of the circumference to the diameter is $\frac{3.1416}{1}$, or 3.1416.

That is, $$\pi = 3.1416.*$$

503. *Exercise.*—By means of the value of π just found and the formulas for the circumference and area of a circle, find the circumference and area of a circle whose radius is 23.16 inches.

* The earliest known attempt to obtain the area of the circle or to "square the circle" is recorded in a MS. in the British Museum recently deciphered. It was written by an Egyptian priest, *Ahmes*, at least as early as 1700 B.C., and possibly several centuries earlier. The method was to *deduct from the diameter of the circle one-ninth of itself and square the remainder.* This is equivalent to using a value of π equal to 3.16. *Archimedes* (about 250 B.C.), the greatest mathematician of ancient times, proved, by methods essentially the same as those employed in the text, that the true value of π lies between $3\frac{1}{7}$ and $3\frac{10}{71}$, i. e., between 3.1429 and 3.1408. *Ptolemy* (about 150 B.C.) used the value 3.1417. In the 16th century *Metrus*, of Holland, using polygons up to 1536 sides, obtained the easily-remembered approximation $\frac{355}{113}$ (write 113355 and divide last three by first three), which is correct to six places of decimals. *Romanus*, also of Holland, using polygons of 1,073,741,324 sides, soon after computed sixteen places. With the better methods of higher mathematics various mathematicians have extended the computations gradually, until *Mr. Shanks*, in 1873, published a result to 707 places, the first 411 of which have been verified by *Dr. Rutherford.* The following are the first figures of his result.
π=3.141,592,653,589,793,238,462,643,383,279,502,884,197,169,399,375,105,8.
How accurate a value this is may be inferred from Prof. Newcomb's remark that *ten* decimals would be sufficient to calculate the circumference of the earth to a fraction of an inch if we had an exact knowledge of the diameter.

The Greeks sought in vain for a perfectly accurate result or geometrical construction for obtaining a square equivalent to the circle, as did many mediæval mathematicians. "Circle squarers" still exist among the ignorant, although *Lambert* (about A.D. 1750) proved π incommensurable, i. e., inexpressible as a finite fraction, and *Lindemann*, in 1882, proved it is also transcendental, *i.e.*, inexpressible as a radical or root of any algebraic equation with integral coefficients.

PROBLEMS OF DEMONSTRATION

504. The angle at the centre of a regular polygon is the supplement of any angle of the polygon.

505. If the sides of a regular circumscribed polygon are tangent to the circle at the vertices of the similar inscribed polygon, then each vertex of the circumscribed figure lies in the prolongation of the apothem of the inscribed.

506. If the sides of a regular circumscribed polygon are tangent to the circle at the middle points of the arcs subtended by the sides of a similar inscribed polygon, then the sides of the circumscribed figure are parallel to those of the inscribed, and the vertices lie in the prolongation of the radii.

507. If from any point within a regular polygon of n sides perpendiculars are drawn to the several sides, the sum of these perpendiculars is equal to n times the apothem.

Hint.—Apply § 495.

508. The area of a circumscribed square is double that of an inscribed square.

509. The side of an inscribed equilateral triangle is equal to one-half the side of a circumscribed equilateral triangle, and the area of the first is one-fourth that of the second.

510. The apothem of an inscribed equilateral triangle is equal to half the radius.

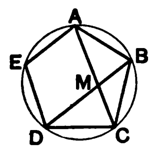
511. The apothem of a regular inscribed hexagon is equal to half the side of the inscribed equilateral triangle.

512. The radius of a regular inscribed polygon is a mean proportional between its apothem and the radius of the similar regular circumscribed polygon.

513. The area of the ring included between two concentric circles is equal to that of a circle whose radius is one half a chord of the outer circle drawn tangent to the inner.

514. In two circles of different radii, angles at the centre subtended by arcs of equal length are to each other inversely as their radii.

515. Two diagonals of a regular pentagon, not drawn from a common vertex, divide each other in extreme and mean ratio.



Hint.—Prove the triangles ABC and BCM similar (§ 275). Then prove $AM = AB = BC$ (§ 77), and substitute in the proportion derived from the first step.

PROBLEMS OF CONSTRUCTION

516. Having given a circle, to construct the circumscribed hexagon, octagon, and decagon.

517. Upon a given straight line as a side to construct a regular hexagon.

518. Having given a circle and its centre, to find two opposite points in the circumference by means of compasses only.

519. To divide a right angle into five equal parts.

520. To inscribe a square in a given quadrant.

521. Having given two circles, to construct a third circle equivalent to their difference.

522. To divide a circle into any number of equivalent parts by circumferences concentric with it.

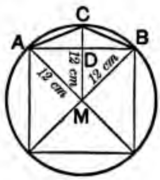
PROBLEMS FOR COMPUTATION

523. (1.) Find the number of degrees in an angle of each of the following regular polygons: (*a*) triangle, (*b*) pentagon, (*c*) hexagon, (*d*) octagon, and (*e*) decagon.

(2.) What is the area of a regular pentagon inscribed in a circle whose radius is 12 cm.?

(3.) If the side of a regular hexagon is 10 m., find the number of square feet in its area.

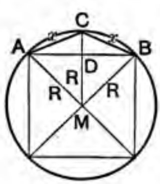
(4.) Find the area of a regular octagon inscribed in a circle whose radius is 12 cm.



(5.) If the radius of a circle is R, find the side and the apothem of a regular inscribed (*a*) triangle, (*b*) square, (*c*) hexagon.

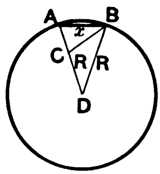
(6.) If, in the above example, $R = 15.762$, find the numerical value of the side and apothem for each of the three polygons.

(7.) Prove that the side of a regular octagon, inscribed in a circle whose radius is R, is equal to $R\sqrt{2-\sqrt{2}}$.



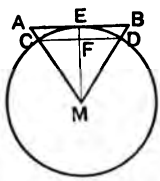
(8.) Find the apothem of a regular octagon inscribed in a circle whose radius is R.

(9.) If the radius of a circle is R, find the side of a regular inscribed decagon.



(10.) What is the apothem of the above decagon?

(11.) Find the side of a regular hexagon circumscribed about a circle whose radius is R.



(12.) If the radius of a circle is R, prove that the area of a regular inscribed dodecagon is $3R^2$.

(13.) There are three regular hexagons; the side of the first is 20 in., that of the second is 1 m., that of the third 5 ft. Find in meters the side of a fourth regular hexagon whose area is equal to the sum of the areas of the first three.

(14.) A wheel, having a radius of 1.5 ft., made 3360 revolutions in going over the road from one town to another. How many miles apart are the towns?

(15.) If the circumference of a circle is 50 in., find the radius.

(16.) If a wheel has 35 cogs, and the distance between the middle points of the cogs is 12 in., find the radius of the wheel.

(17.) Find the width of a ring of metal the outer circumference of which is 88 m. in length, and the inner circumference 66 m.

(18.) If the radius of a circle is 16 cm., how many degrees, minutes, and seconds are there in an arc 10 cm. long?

(19.) Find the number of feet in an arc of 20° if the radius of the circle is 12 m.

(20.) How many degrees are there in an arc whose length is equal to the radius of the circle?

(21.) If an arc of $30^\circ = 12.5664$ in., find the radius of the circle.

(22.) If the radius of a circle is 15 cm., find the length of the arc subtended by a chord 15 cm. in length.

(23.) If the circumference of a circle is c, find its radius and diameter.

(24.) Find the area of a circle whose radius is (*a*) 11 in.; (*b*) 17.146 m.; (*c*) 35 ft.

(25.) Find the ratio of the areas of two circles if the radius of one is the diameter of the other.

(26.) If the circumference of a circle is 60 ft., find the area.

(27.) The radius of a circle is 13 in. Find the side of a square whose area is equal to that of the circle.

(28.) The side of an inscribed square is 23 m. What is the area of the circle?

(29.) What is the area of a circle inscribed in a square whose surface contains 211 ares?

(30.) Find the side of the largest square that can be cut from the cross-section of a tree 14 ft. in circumference.

(31.) If the diameter of a given circle is 5 cm., find the diameter of a circle one-fourth as large.

(32.) A rectangle and a circle have equal perimeters. Find the difference in their areas if the radius of the circle is 9 in. and the width of the rectangle is three-fourths its length.

(33.) If the radius of a circle is 25 m., what is the radius of a concentric circle which divides it into two equivalent parts?

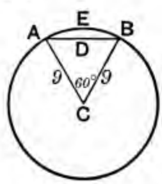
(34.) The radii of two concentric circles are respectively 9 and 6 in. Find the area of the ring bounded by their circumferences.

(35.) The chord of a segment of a circle is 34 in. in length, and the height of the segment is 8 in. Find the radius.

(36.) In a circle whose radius is 18 in., find the height of a segment whose chord is 28 in. in length.

(37.) If the radius of a circle is 16 cm., what is the area of a sector having an angle of 24°?

(38.) The radius of a circle is 9 in. Find the area of a segment whose arc is 60°.



Hint.—Area of segment $AEBD$ = area of sector $AEBC$ minus area of triangle ABC.

(39.) If the radius of a circle is R, find the area of the segment subtended by the side of a regular hexagon.

(40.) If the radius of a circle is R, find the area of a segment subtended by the side of (*a*) an inscribed equilateral triangle, (*b*) an inscribed regular octagon, (*c*) an inscribed regular decagon.

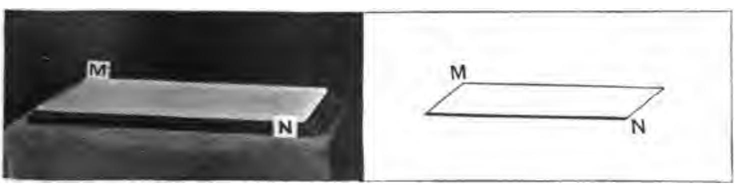
GEOMETRY OF SPACE

BOOK VI

STRAIGHT LINES AND PLANES

524. *Def.*—A **plane** has already been defined as "a surface such that, if any two points in it are taken, the straight line passing through them lies wholly in the surface." § 8

A plane is regarded as indefinite in extent, but is usually represented to the eye by a parallelogram lying in it.



525. *Def.*—A plane is **determined** by given conditions, if it is the only plane fulfilling these conditions.

PROPOSITION I. THEOREM

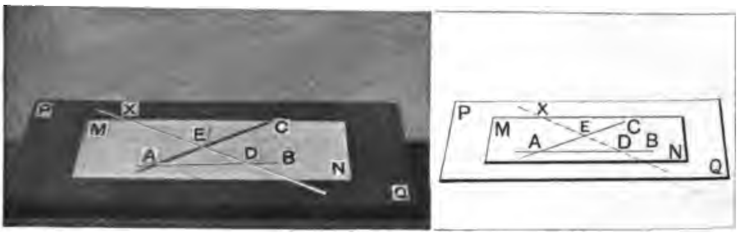
526. *A plane is determined if it passes through:*

I. *Three points not in the same straight line.*

II. *A straight line and a point without that line.*

III. *Two intersecting straight lines.*

IV. *Two parallel straight lines.*



I. GIVEN—three points, A, B, and C not in the same straight line.

TO PROVE—that one and only one plane can be passed through them.

Pass a plane MN through one of the points, turn it about this point until it contains one of the other points, and then turn it about these two points until it contains the third.

No other plane will contain these points.

For, suppose PQ to be such a plane.

Take X *any point* in PQ. We will prove it also lies in MN.

Draw the straight lines AB and AC.

These will be in both planes, since A, B, and C lie in both planes. § 524

Through X draw a straight line in PQ cutting AB and AC in D and E.

Since D and E lie in the plane MN, the straight line DEX lies wholly in MN. § 524

Hence X, a point in DE, lies in the plane MN.

Thus *any* point, that is, *every* point in the plane PQ lies also in the plane MN, and in like manner we can prove that every point in MN lies in PQ.

Therefore the two planes coincide. Q. E. D.

II. GIVEN—the straight line AB and the point C without AB.

TO PROVE—that one and only one plane can be passed through them.

The plane passed through C and any two points of AB will contain AB. § 524

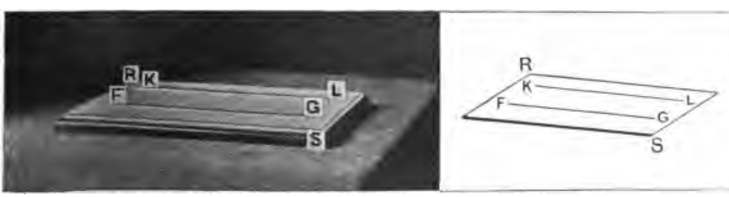
We can pass no other plane through AB and C, for then we would have two planes containing three points not in the same straight line, which is impossible. Q. E. D.

III. GIVEN—the straight lines AB and AC intersecting in A.

TO PROVE—that one and only one plane can be passed through them.

The plane passed through the three points A, B, and C will contain the straight lines AB and AC. § 524

We can pass no other plane through AB and AC, for then we would have two planes containing three points not in the same straight line, which is impossible. Q. E. D.



IV. GIVEN—the parallel straight lines FG and KL.

TO PROVE—that one and only one plane can be passed through them.

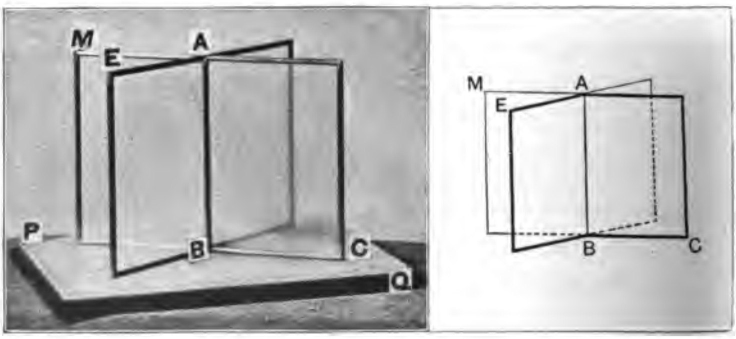
By definition these parallel lines lie in the same plane. § 31

There cannot be two planes passed through them, for then we would have two planes containing three points F, G, and K, not in the same straight line, which is impossible. Q. E. D.

527. *Def.*—The **intersection** of two planes is the line common to both planes.

PROPOSITION II. THEOREM

528. *If two planes intersect, their intersection is a straight line.*



GIVEN two intersecting planes, MB and EB.

TO PROVE their intersection is a straight line.

If possible, suppose the intersection is not straight.

It would then contain three points not in the same straight line.

That is, the two planes would contain three points not in the same straight line, which is impossible. § 526 I

Therefore the intersection must be a straight line. Q. E. D.

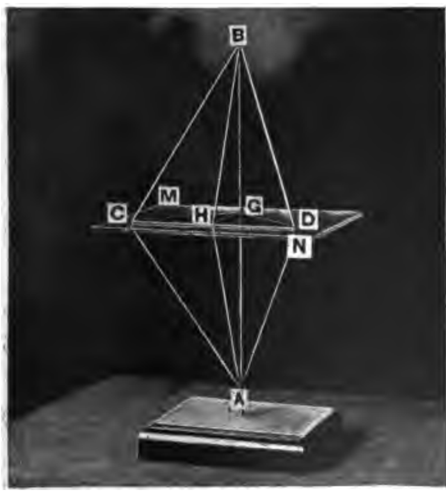
PERPENDICULAR AND OBLIQUE LINES AND PLANES

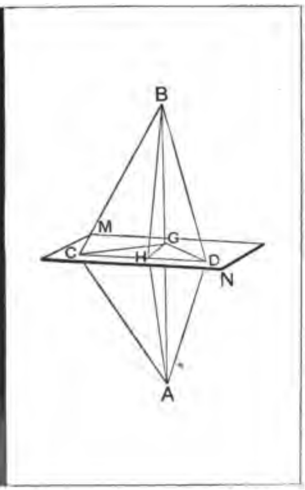
529. *Def.*—If a straight line meet a plane, its point of meeting is called its **foot.**

530. *Defs.*—A straight line is **perpendicular** to a plane, if it is perpendicular to every straight line in the plane drawn through its foot. In the same case the plane is said to be perpendicular to the line.

PROPOSITION III. THEOREM

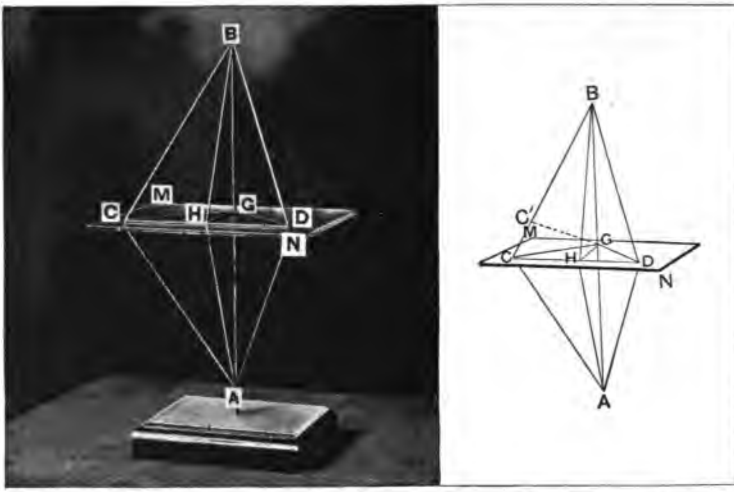
531. *If two intersecting straight lines are perpendicular to a third at the same point, their plane is perpendicular to that straight line.*





GIVEN—the two intersecting straight lines GC and GD perpendicular to the straight line BG at the point G.

TO PROVE—that the plane MN passed through GC and GD is perpendicular to BG.



In the plane MN draw through G any straight line GH.

Let CD be any straight line cutting the lines GC, GH, and GD in C, H, and D.

Produce the line BG to A making GA equal to GB, and join A and B to C, H, and D.

Then, since GC is perpendicular to BA at its middle point,

$$CB = CA. \qquad \S\ 103$$

Similarly $$DB = DA.$$

Hence the triangles BCD and ACD are equal. § 89

And the homologous angles BCH and ACH are equal.

Hence the triangles BCH and ACH are equal. § 79

Therefore their homologous sides BH and AH are equal.

Therefore GH is perpendicular to BA. § 104

But GH is *any* straight line in MN passing through G.

Therefore *every* straight line in MN passing through G is perpendicular to BA; that is, MN is perpendicular to BA. § 530

Q. E. D.

532. COR. I. *At a given point in a straight line one and only one plane can be drawn perpendicular to that straight line.*

Hint.—Let AB be the straight line and G the point.

At G draw the straight lines GC and GD perpendicular to AB.

The plane of these lines will be perpendicular to AB. (Why?)

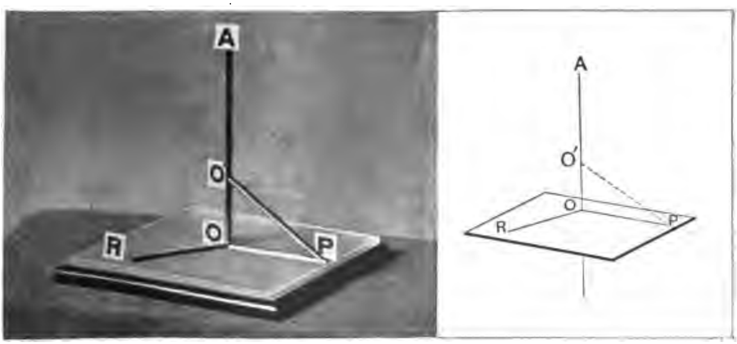
Only one such plane can be drawn.

For any other plane passing through G cannot contain both of the lines GC and GD. (Why?)

It must therefore cut one of the planes BGC and BGD, say BGC, in some line GC' other than GC and GD.

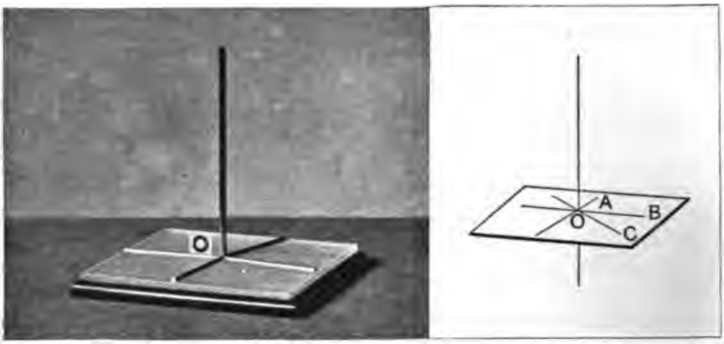
Since BGC' is not a right angle, this second plane is not perpendicular to AB. (Why?)

533. COR. II. *Through a given point without a straight line one and only one plane can be passed perpendicular to that straight line.*



Hint.—Use § 531 to draw *one* such plane. Any other plane cuts AO either at O or at some other point, O'. § 532 shows that the first is not perpendicular. Show also that the second is not.

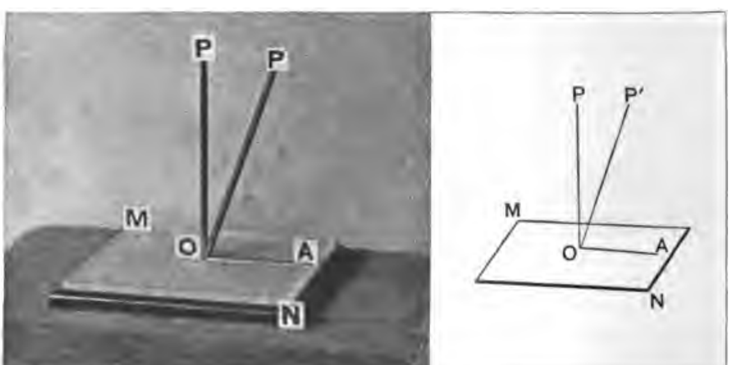
534. COR. III. *All the perpendiculars to a given straight line at the same point lie in a plane perpendicular to that line at that point.*



Hint.—Every pair of these perpendiculars, as OA and OB, determines a plane perpendicular at O. (Why?)

And all the planes thus determined must coincide. (Why?) Hence, etc.

535. COR. IV. *At a point in a plane one and only one perpendicular to the plane can be drawn.*



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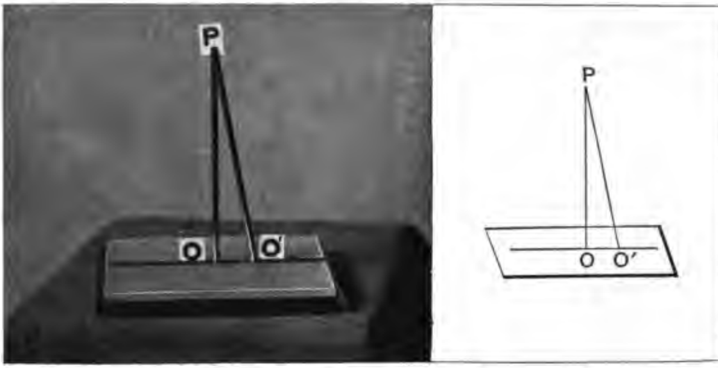
537. COR. *From a point without a plane one and only one perpendicular to the plane can be drawn.*



Hint.—Apply the Proposition and § 34.

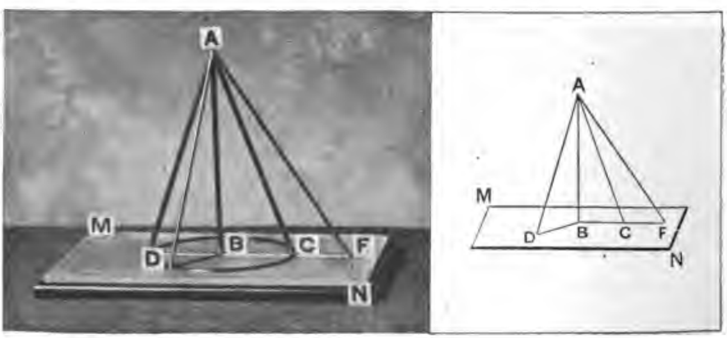
538. *Def.*—By the **distance from a point to a plane** is meant the shortest distance, and therefore the perpendicular distance.

PROPOSITION V. THEOREM

539. *If oblique lines are drawn from a point to a plane:*

I. *Those meeting the plane at equal distances from the foot of the perpendicular are equal.*

II. *Of two unequally distant, the more remote is the greater.*



I. GIVEN—the oblique lines AC and AD meeting the plane MN at the equal distances BC and BD from the perpendicular AB.

TO PROVE $AC=AD$.

In the triangles ABC and ABD, AB is common; $BC=BD$ by hypothesis; and the angles ABC and ABD are equal, being right angles.

Therefore the triangles are equal, and $AC=AD$. § 79

Q. E. D.

II. GIVEN—the oblique lines AF and AD meeting MN so that

$$BF>BD$$

TO PROVE $AF>AD$

On BF take $BC=BD$ and draw AC.

Then, from *plane* geometry, $AF>AC$. § 99

But $AD=AC$. § 16 I

Therefore $AF>AD$. Q. E. D.

540. COR. Conversely:

I. *Equal oblique lines from a point to a plane meet the plane at equal distances from the foot of the perpendicular.*

II. *Of two unequal oblique lines the greater meets the plane at the greater distance from the foot of the perpendicular.*

Hint.—Prove as in § 100.

541. *Remark.*—Article 540 supplies practical methods of drawing a straight line perpendicular to a plane, as a floor or a blackboard.

I. *To erect a perpendicular to a plane at a given point in it.*

With the given point as centre, describe a circumference in the given plane.

Take three strings of equal length somewhat longer than the radius of the circumference.

To each of three points on the circumference attach an end of one string.

Unite the three remaining ends in a knot and pull the strings taut.

A line through the given point and the knot is the perpendicular required. Prove the method correct by supposing if possible that the foot of the perpendicular from the knot is not in the given point, and apply § 103.

II. *To draw a perpendicular to a given plane from a given point without it.*

From the point with a string of convenient length measure three equal distances to the plane.

The centre of the circumference which passes through the three points thus found is the foot of the required perpendicular. (Why?)

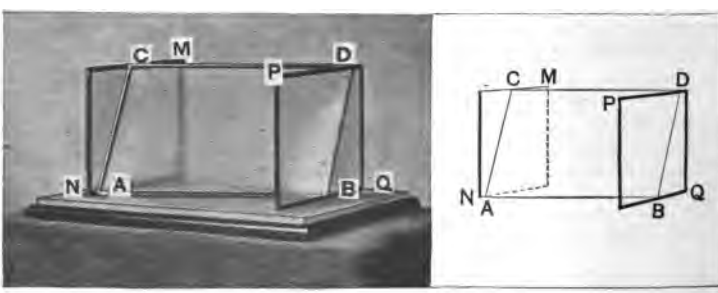
PARALLEL LINES AND PLANES

542. *Def.*—A straight line and a plane are **parallel** to each other if they cannot meet, however far produced.

543. *Def.*—Two planes are **parallel** to each other if they cannot meet, however far produced.

PROPOSITION VI. THEOREM

544. *If two parallel planes are cut by a third plane, their intersections with this plane are parallel.*



GIVEN—the parallel planes *MN* and *PQ* cut by the plane *AD* in the lines *AC* and *BD*.

TO PROVE *AC* and *BD* parallel.

Since the planes *MN* and *PQ* cannot meet, the lines *AC* and *BD* lying in them cannot meet.

Moreover these lines lie in the same plane *AD*.

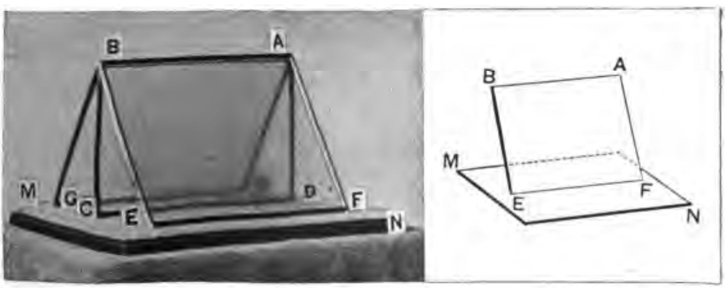
Therefore they are parallel. § 31

Q. E. D.

545. COR. *Parallel lines AB and CD intercepted between parallel planes are equal.*

PROPOSITION VII. THEOREM

546. *If a straight line is parallel to a plane, the intersection of the plane with a plane passed through the line is parallel to the line.*



GIVEN—the line *BA* parallel to the plane *MN* and a plane *BF* passing through *BA* and intersecting *MN* in *EF*.

TO PROVE *BA* parallel to *EF*.

These lines lie in the same plane.

They cannot meet, for *BA* cannot meet the plane *MN* in which *EF* lies.

Therefore they are parallel. § 31

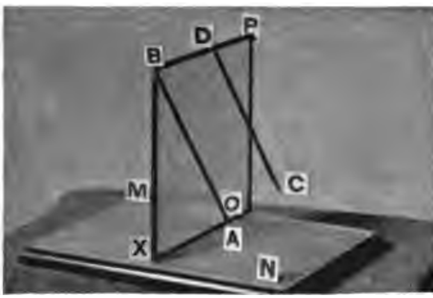
Q. E. D.

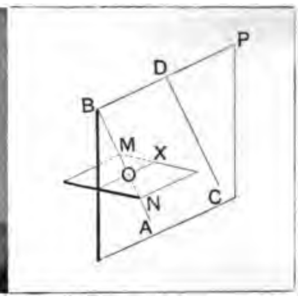
547. COR. *If two intersecting straight lines are parallel to a plane, their plane is parallel to the given plane.*

Hint.—If their plane were not parallel to the given plane it would intersect it in a line which would be parallel to both the given lines.

PROPOSITION VIII. THEOREM

548. *A plane which cuts one of two parallel lines must, if sufficiently produced, cut the other also.*





GIVEN—the parallel lines AB and CD, one of which, AB, is cut by the plane MN in the point O.

TO PROVE that CD is also cut by MN.

Pass a plane through AB and CD.

As this plane and the plane MN have the point O in common, their intersection must contain O. Call it OX.

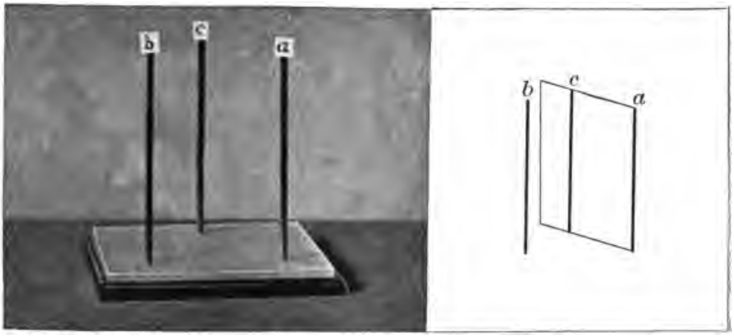
Now suppose, if possible, that MN does not cut the line CD, but is parallel to it.

Then OX will also be parallel to CD. § 546

And there will be *two* lines, OX and OA through O, parallel to CD, which is impossible.

Therefore MN *must* cut CD. Q. E. D.

549. COR. I. *If two straight lines a and c are parallel to a third b, they are parallel to each other.*

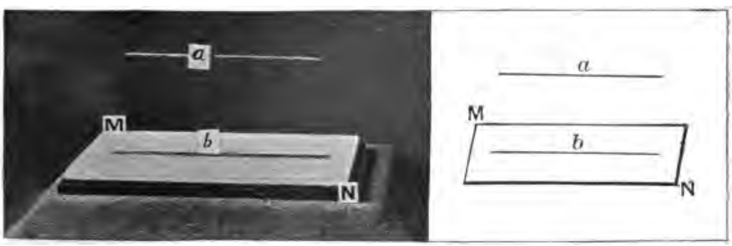


Hint.—Pass a plane through *a* and any point of *c*.

This plane will entirely contain *c*. Otherwise it would cut *c* and therefore *b*, which is parallel to *c*, and also *a*, which is parallel to *b*. This contradicts the hypothesis that it contains *a*.

Prove also that *a* and *c* cannot meet.

550. COR. II. *If two straight lines a and b are parallel, any plane MN, that contains one, as b, and not the other, is parallel to the second.*



Hint.—If *MN* is not parallel to *a*, it will cut it.

This is impossible, for then *MN* would cut *b* also.

Therefore *MN* is parallel to *a*.

551. COR. III. *If two intersecting straight lines are parallel to two other intersecting straight lines, the plane of the first pair is parallel to the plane of the second pair.*

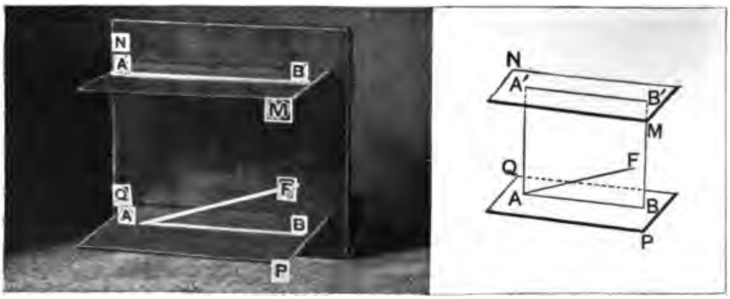
Hint.—Apply § 550 and then § 547.

PROPOSITION IX. THEOREM

552. *If two planes are parallel:*

I. *Any straight line that cuts one cuts the other.*

II. *Any plane that cuts one cuts the other.*



I. GIVEN—the parallel planes MN and PQ and the straight line AF cutting PQ in the point A.

TO PROVE—that AF is not parallel to MN but cuts MN.

Through AF and any point A' of MN not in AF pass a plane $A'B$.

Since this plane has a point in common with each of the parallel planes, it will intersect each in straight lines AB and $A'B'$.

These lines will be parallel. § 544

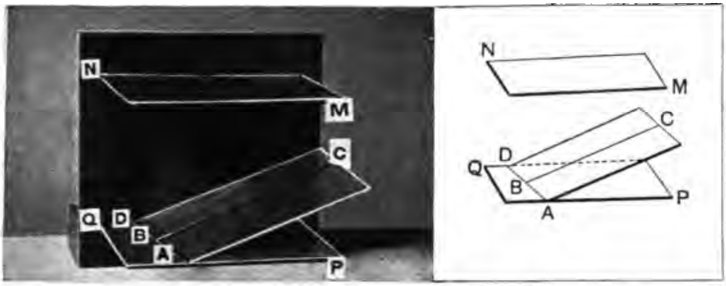
In the plane $A'B$ we have AF cutting AB, one of the two parallels AB and $A'B'$.

It therefore cuts the other, $A'B'$, since AF and AB cannot both be parallel to $A'B'$. Ax. *b*

Therefore AF cutting $A'B'$ cuts the plane MN in which $A'B'$ lies. Q. E. D.

II. GIVEN—the plane CD intersecting PQ in the straight line AD.

TO PROVE that CD also intersects MN.



In the plane CD draw any straight line BC cutting AD.

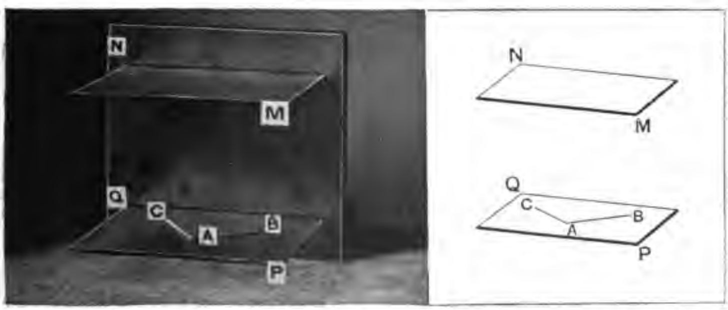
This line cuts PQ, and therefore cuts MN, by the first part of the proposition.

Therefore the plane CD, in which BC lies, will cut MN.

Q. E. D.

553. COR. I. *If two planes are parallel to a third plane they are parallel to each other.*

554. COR. II. *Through a given point without a given plane there can be drawn a plane parallel to the given plane, and but one.*



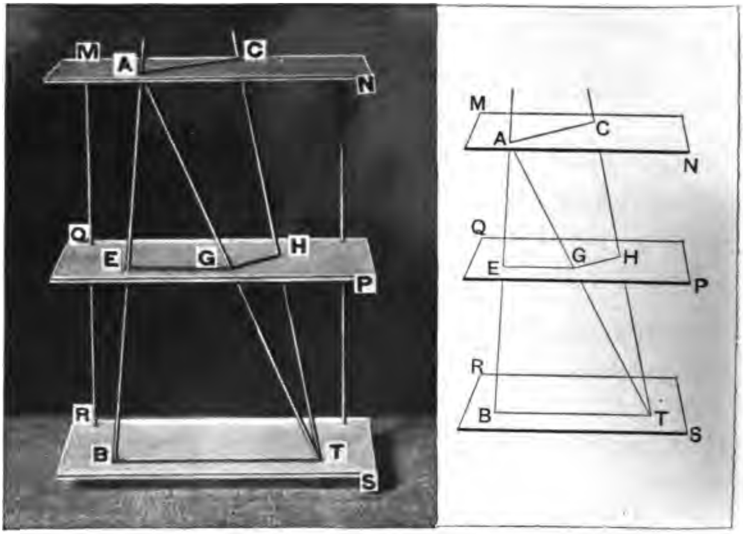
Hint.—Through the point A, without the plane MN, draw two straight lines AB and AC parallel to MN.

PQ, the plane of AB and AC, will be parallel to MN.

No other plane through A could be parallel to MN, for it would cut PQ, and therefore also MN.

PROPOSITION X. THEOREM

555. *If two straight lines are cut by three parallel planes, their corresponding segments are proportional.*



GIVEN—the straight lines AB and CT cut by the parallel planes MN, PQ, and RS in the points A, E, B, and C, H, T.

TO PROVE $\dfrac{AE}{EB}=\dfrac{CH}{HT}$.

Join A to T by a straight line cutting PQ in G.

Draw EG, BT, GH, and AC.

Then EG and GH will be parallel to BT and AC respectively. § 544

Therefore $\dfrac{AE}{EB}=\dfrac{AG}{GT}$, and $\dfrac{AG}{GT}=\dfrac{CH}{HT}$. § 271

Hence $\dfrac{AE}{EB}=\dfrac{CH}{HT}$. Ax. 1

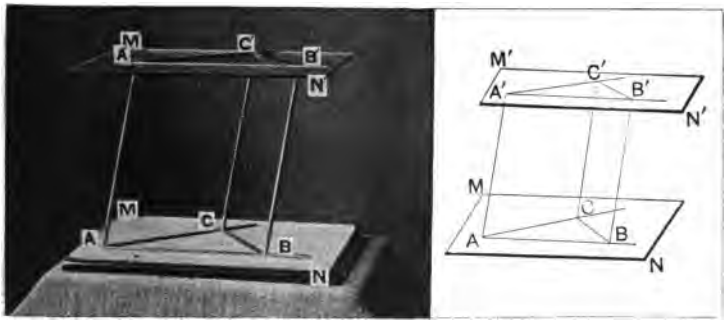
Q. E. D.

556. COR. *If a series of lines passing through a common point are cut by two parallel planes, their corresponding segments are proportional.*

Hint.—Pass a third plane through the common point parallel to one (and hence the other) of the two given planes.

PROPOSITION XI. THEOREM

557. *If two angles not in the same plane have their sides respectively parallel and extending from their vertices in the same direction, they are equal.*



GIVEN—the angles BAC and $B'A'C'$, whose sides, AB, $A'B'$, and AC, $A'C'$, are respectively parallel and extending in the same direction.

TO PROVE angle BAC=angle $B'A'C'$.

Take $AB=A'B'$ and $AC=A'C'$ and join AA', BB', CC'.

Then AB' and AC' will be parallelograms. § 126

Hence BB' and CC' are equal to and parallel to AA'. §§ 117, 114

Hence BB' and CC' are equal to and parallel to each other. Ax. 1, § 549

Therefore BC' is a parallelogram, and $BC=B'C'$. § 126

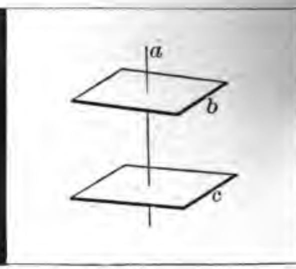
The triangles ABC and $A'B'C'$ are therefore equal. § 89

Hence angle BAC=angle $B'A'C'$. Q. E. D.

558. COR. *If two angles not in the same plane have their sides respectively parallel and extending in opposite directions from their vertices, they are equal; if two corresponding sides extend in the same direction, and the other two in opposite directions, the angles are supplementary.*

PROPOSITION XII. THEOREM

559. *If two planes are perpendicular to the same straight line, they are parallel.*



GIVEN—the planes b and c perpendicular to the straight line a.

TO PROVE b and c parallel.

If they should meet, we should have through any point of their intersection two planes, b and c, perpendicular to the same straight line a.

This is impossible. § 533

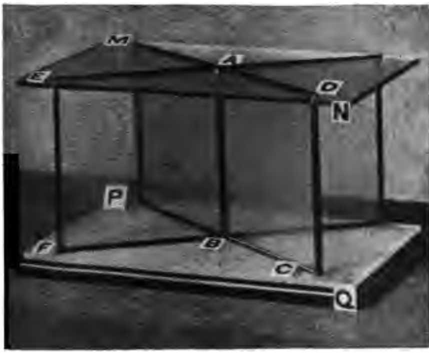
Therefore b and c are parallel. Q. E. D.

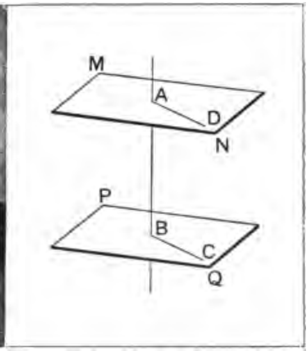
560. *Exercise.*—Prove this proposition as a consequence of §§ 33, 551.

Hint.—Pass two planes through a intersecting b and c in straight lines perpendicular to a.

PROPOSITION XIII. THEOREM

561. *If a straight line is perpendicular to one of two parallel planes, it is perpendicular to the other.*





GIVEN—the parallel planes MN and PQ, and the line AB perpendicular to MN at A.

TO PROVE AB perpendicular to PQ.

Since AB cuts MN, it also cuts PQ in some point B. § 552 I

[If two planes are parallel, any line that cuts one cuts the other.]

Through B draw in PQ any straight line BC, and through AB and BC pass a plane intersecting MN in AD.

Then BC is parallel to AD. § 544

[If two planes are parallel, their intersections with a third plane are parallel.]

But AB is perpendicular to AD. § 530

[A straight line perpendicular to a plane is perpendicular to every straight line drawn in the plane through its foot.]

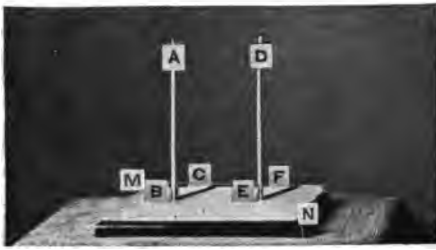
Therefore AB is also perpendicular to BC. § 36

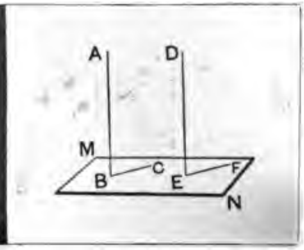
Since AB is perpendicular to *any* straight line drawn in PQ through B, it is perpendicular to PQ. § 530

Q. E. D.

PROPOSITION XIV. THEOREM

562. *If a plane is perpendicular to one of two parallel lines, it is perpendicular to the other.*





GIVEN—the parallel lines AB and DE and the plane MN perpendicular to AB at B.

TO PROVE MN perpendicular to DE.

Since MN cuts AB, it also cuts DE in some point E. § 548

[If two lines are parallel, any plane that cuts one cuts the other.]

Through E draw in MN any straight line EF, and through B draw in MN the line BC parallel to EF.

Then angle DEF = angle ABC. § 557

But, since BC lies in MN, ABC is a right angle. § 530

Hence DEF is a right angle.

Since *any* straight line in MN through E is perpendicular to DE, MN is perpendicular to DE. Q. E. D.

563. COR. I. *If two straight lines are perpendicular to the same plane, they are parallel.*

Hint.—Suppose AB and DE perpendicular to MN.

Through any point of DE draw a line, as ED', parallel to AB.

Prove that DE and ED' coincide.

564. *Exercise.*—Prove § 549 by means of §§ 562, 563.

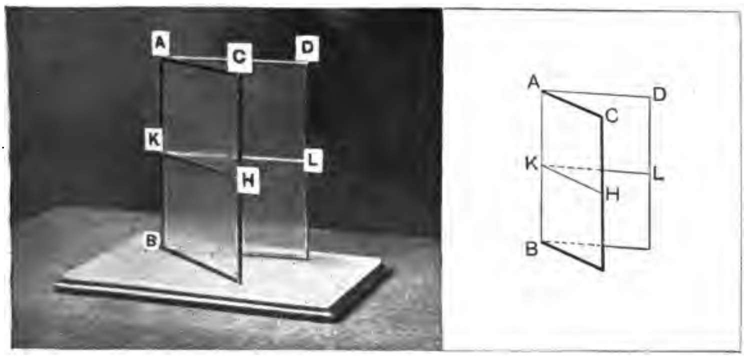
565. COR. II. *The perpendicular distance between two parallel planes is everywhere the same.*

DIEDRAL ANGLES AND PROJECTIONS

566. *Defs.*—When two planes meet and are terminated at their common intersection, they are said to form a **diedral angle.**

The planes are called the **faces** of the diedral angle, and their intersection, the **edge.**

The faces are regarded as indefinite in extent.



We may designate a diedral angle by two points on its edge and one other point in each face, the former two being written between the latter two.

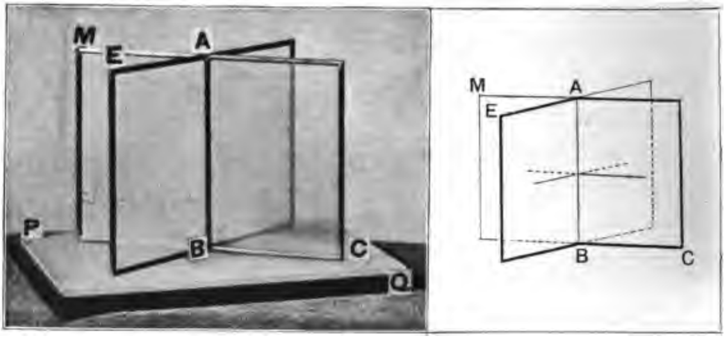
Thus, in the figure, the two planes BC and BD meeting in the line AB form the diedral angle $CABD$; BC and BD are the faces of the diedral angle, and AB is its edge.

If there is only one diedral angle at an edge, it may be designated by two points on its edge; thus the diedral angle $CABD$ may also be called the diedral angle AB.

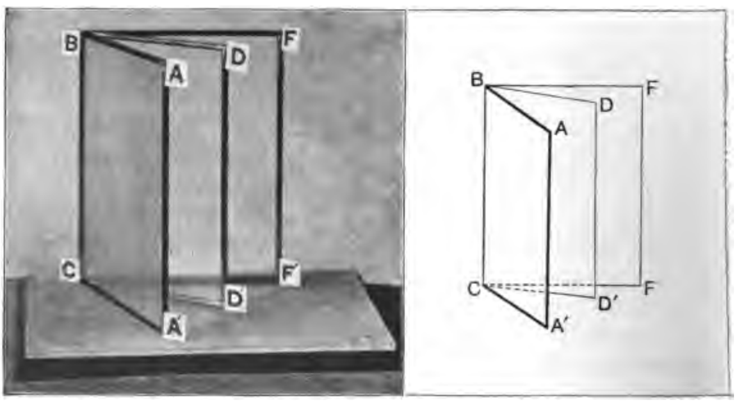
567. *Def.*—The **plane angle** of a diedral angle is the angle formed by two straight lines drawn one in each face of the diedral angle perpendicular to its edge at the same point.

Thus HKL is the plane angle of the diedral angle $CABD$.

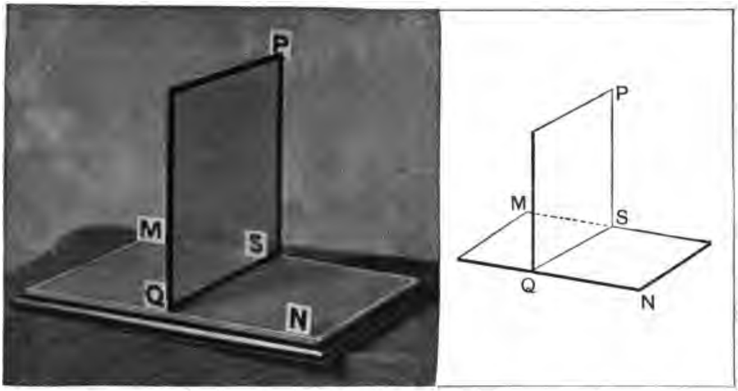
568. *Def.*—Two diedral angles are **vertical** if the faces of one are the prolongations of the faces of the other.



569. *Def.*—Two diedral angles are **adjacent** when they have a common edge and a common face lying between them; as $ABCD$ and $FBCD$.



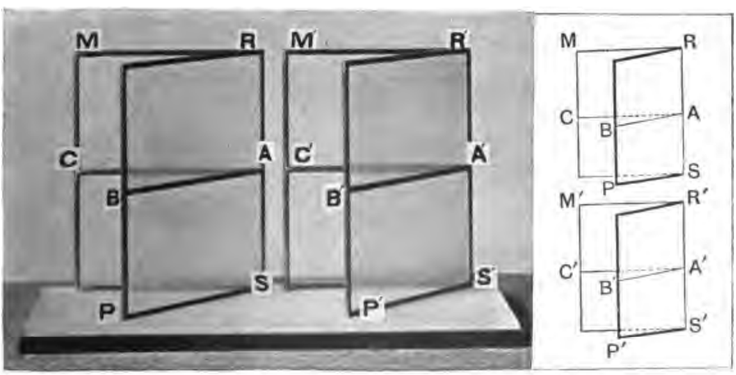
570. *Def.*—If a plane meets another plane so as to form with it two equal adjacent diedral angles, each of these diedral angles is called a **right diedral angle**, and the first plane is said to be **perpendicular** to the second.



Thus the plane PQ is perpendicular to the plane MN, if the diedral angles $PQSN$ and $PQSM$ are equal.

PROPOSITION XV. THEOREM

571. *If two diedral angles are equal, their plane angles are equal.*



GIVEN the equal diedral angles $MRSP$ and $M'R'S'P'$.

TO PROVE their plane angles CAB and $C'A'B'$ equal.

Superpose the diedral angle $M'R'S'P'$ upon its equal $MRSP$, letting A' fall at A.

Then, since $A'B'$ and AB are both perpendicular to the line RS at A in the plane RP, they coincide. § 18

Similarly $A'C'$ and AC coincide.

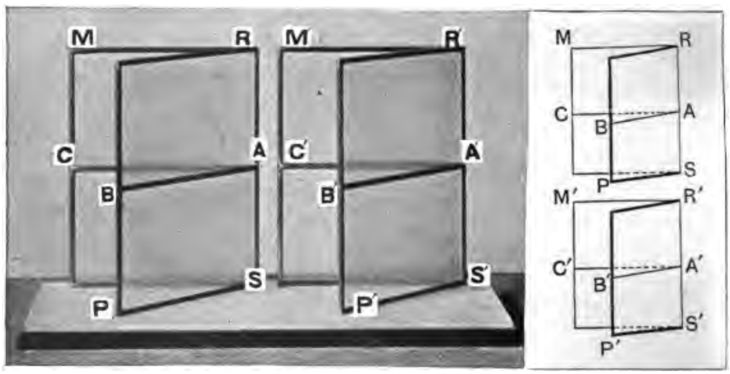
Therefore the angles CAB and $C'A'B'$ are equal. § 15

Q. E. D.

PROPOSITION XVI. THEOREM

572. *If the plane angles of two diedral angles are equal, the diedral angles are equal.*

[Converse of Proposition XV.]



GIVEN—two diedral angles, $MRSP$ and $M'R'S'P'$, whose plane angles, CAB and $C'A'B'$, are equal.

TO PROVE the diedral angles equal.

Since RS is perpendicular to the lines AB and AC, it is perpendicular to their plane. § 531

Similarly $R'S'$ is perpendicular to the plane of $A'B'$ and $A'C'$.

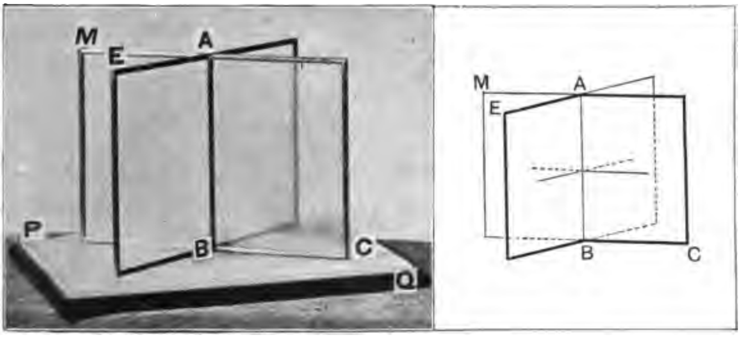
Place the angle $C'A'B'$ upon its equal CAB.

Then $R'S'$ will coincide with RS. § 535

Hence the planes $R'P'$ and RP will coincide. § 526
Similarly the planes $M'S'$ and MS will coincide.
The diedral angles are therefore equal. § 15

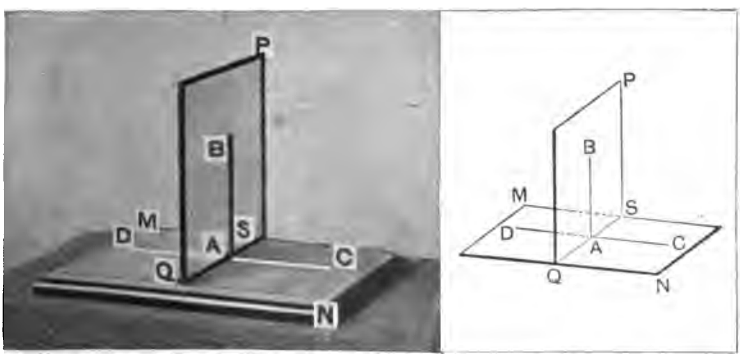
Q. E. D.

573. COR. *Two vertical diedral angles are equal.*



PROPOSITION XVII. THEOREM

574. *If a straight line is perpendicular to a plane, every plane passed through the line is perpendicular to the plane.*



GIVEN—the straight line AB perpendicular to the plane MN at A, and the plane PQ passed through AB intersecting MN in QS.

TO PROVE PQ perpendicular to MN.

Through A draw in MN the line CD perpendicular to QS.

Since AB is perpendicular to MN, it is perpendicular to QS and CD in MN. § 530

Hence BAC and BAD are right angles, and are the plane angles of the diedral angles $PQSN$ and $PQSM$. §§ 16, 567

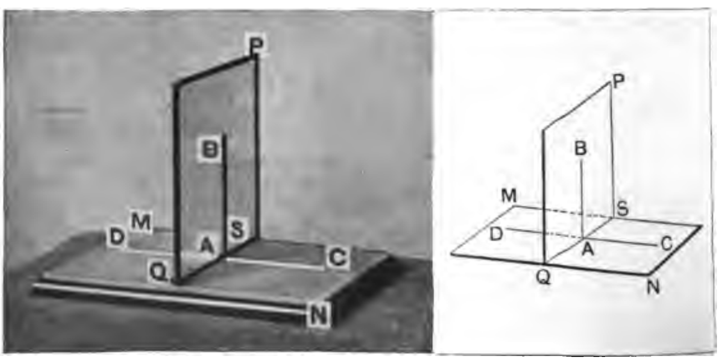
Therefore these diedral angles are equal. § 572

That is, PQ is perpendicular to MN. § 570

Q. E. D.

PROPOSITION XVIII. THEOREM

575. *If two planes are perpendicular to each other, a straight line drawn in one, perpendicular to their intersection, is perpendicular to the other.*



GIVEN—the plane PQ perpendicular to the plane MN and intersecting MN in QS. Draw AB in PQ perpendicular to QS at A.

TO PROVE AB perpendicular to MN.

Through A draw in MN the line CD perpendicular to QS.

Then BAC and BAD will be the plane angles of the equal diedral angles $PQSN$ and $PQSM$. § 567

Hence angle BAC = angle BAD. § 571

Therefore AB is perpendicular to CD. § 16

Since AB is perpendicular to CD and also to QS, it is perpendicular to MN. § 531

Q. E. D.

576. COR. I. *If two planes are perpendicular to each other, a straight line drawn perpendicular to one at any point of their intersection lies in the other.*

Hint.—In the foregoing figure let AB now be drawn perpendicular to MN at the point A of QS.

Then draw AB' in PQ perpendicular to QS.

Prove AB and AB' coincide.

577. COR. II. *If two planes are perpendicular to each other, a straight line drawn from any point of one perpendicular to the other lies in the first.*

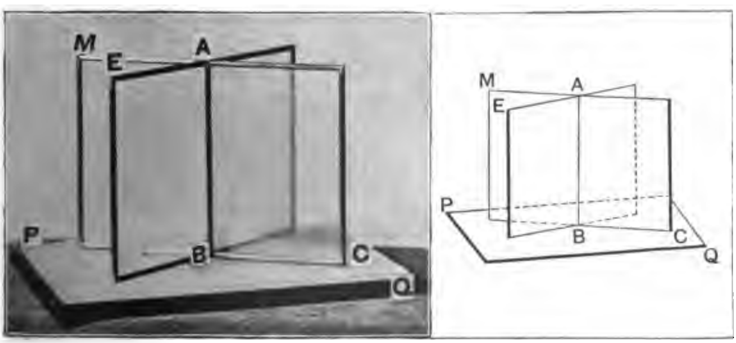
Hint.—In the foregoing figure let BA now be drawn perpendicular to MN from the point B in PQ.

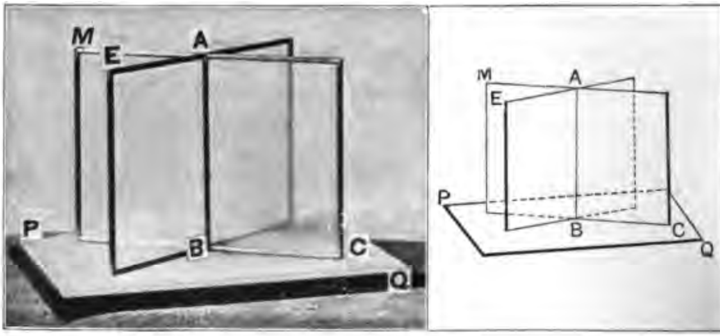
Then draw BA' in PQ perpendicular to QS.

Prove BA and BA' coincide.

PROPOSITION XIX. THEOREM

578. *If two intersecting planes are perpendicular to a third plane, their intersection is perpendicular to that plane.*





GIVEN—the planes MC and EB perpendicular to the plane PQ and intersecting in AB.

TO PROVE AB perpendicular to PQ.

Through any point of AB draw a straight line perpendicular to PQ.

This line will lie in both MC and EB. §§ 576, 577

It must therefore coincide with their intersection AB.

Therefore AB is perpendicular to PQ. Q. E. D.

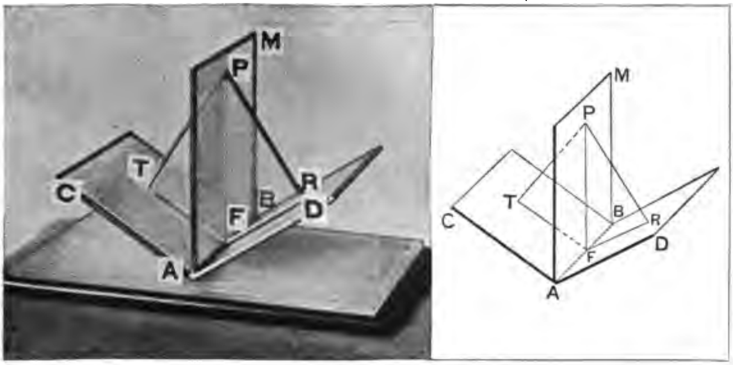
PROPOSITION XX. THEOREM

579. *Every point in the plane that bisects a diedral angle is equally distant from the faces of that angle.*

GIVEN—the plane MA bisecting the diedral angle $DABC$. Let P be any point in MA, and let PT and PR be the perpendiculars dropped from P to the faces BC and BD of the diedral angle.

TO PROVE $PT = PR$.

Through PT and PR pass a plane intersecting the planes BC, BD, and MA in FT, FR, and FP respectively.



Since the line PT is perpendicular to the plane BC, the plane PRT is perpendicular to the plane BC. § 574

Similarly the plane PRT is perpendicular to the plane BD.

Therefore PRT is perpendicular to their intersection AB. § 578

Hence AB is perpendicular to FT, FP, and FR. § 530

Hence PFT and PFR are the plane angles of the equal diedral angles $MABC$ and $MABD$. § 567

Therefore angle PFT=angle PFR. § 571

Consequently the right triangles PTF and PRF are equal. § 85

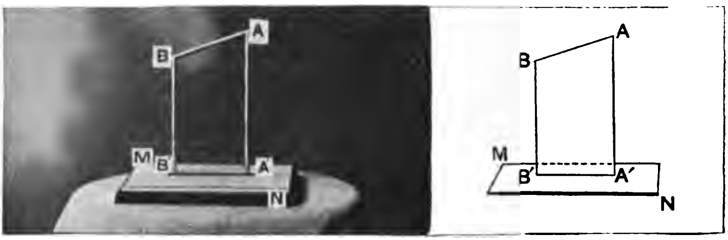
Therefore $PT=PR$. Q. E. D.

580. COR. *The locus of all points within a diedral angle equally distant from its faces is the plane which bisects the diedral angle.*

Hint.—It has been proved that all points in the bisecting plane possess the required property. It only remains to prove that any point outside does not, or that any point which possesses the required property must lie in AM. Let P' be such a point. Pass a plane through P' and the edge AB, and make constructions analogous to those in the preceding figure. Then prove that the plane $P'AB$ must bisect the diedral angle.

PROPOSITION XXI. THEOREM

581. *Through any straight line a plane can be passed perpendicular to any plane; and only one such plane can be drawn unless the given line is itself perpendicular to the given plane.*



GIVEN the straight line AB and the plane MN.

TO PROVE—a plane can be drawn through AB perpendicular to MN.

From any point B of AB drop a perpendicular BB' to MN.

The plane passed through AB and BB' will be perpendicular to MN. § 574

Hence *one* plane can be passed through AB perpendicular to MN.

Now no other plane can be passed through AB perpendicular to MN unless AB is perpendicular to MN.

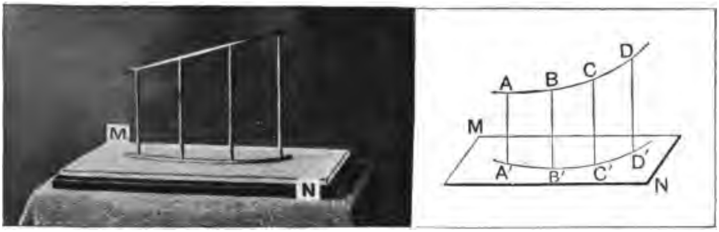
For this other plane would contain BB'. § 577

And would coincide with the first plane, since both contain the intersecting lines AB and BB'. § 526 III

Q. E. D.

582. *Def.*—The **projection of a point upon a plane** is the foot of the perpendicular drawn from the point to the plane.

Thus A' is the projection of the point A upon the plane MN.

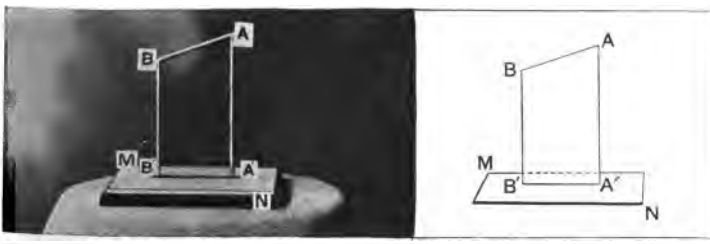


583. *Def.*—The **projection of a line upon a plane** is the locus of the projections of its points.

Thus $A'B'C'D'$ is the projection of $ABCD$ upon MN.

PROPOSITION XXII. THEOREM

584. *The projection of a straight line upon a plane is a straight line.*



GIVEN—the projection $A'B'$ of the straight line AB upon the plane MN.

TO PROVE $A'B'$ a straight line.

Through AB pass a plane perpendicular to MN.

The perpendiculars drawn from the various points of AB to MN must lie in this plane. § 577

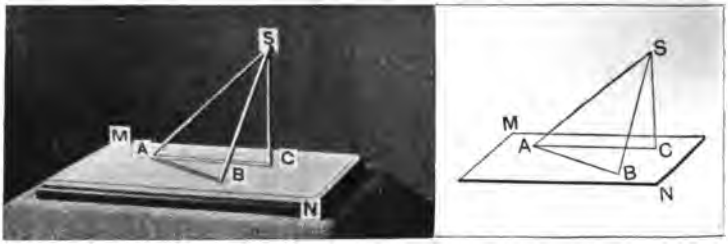
Hence their feet will lie in the intersection of MN with this plane.

Therefore $A'B'$ must be the intersection and is a straight line. § 528

Q. E. D.

PROPOSITION XXIII. THEOREM

585. *The acute angle which a straight line makes with its own projection upon a plane is the least angle which it makes with any line in that plane.*



GIVEN—the straight line AS, its projection AC upon the plane MN, and AB any other straight line in MN through A.

TO PROVE angle $SAC <$ angle SAB.

Take $AB = AC$ and draw SC and SB.

Then the triangles SAC and SAB have two sides of one equal respectively to two sides of the other.

But the third side SC of one is less than the third side SB of the other. § 536

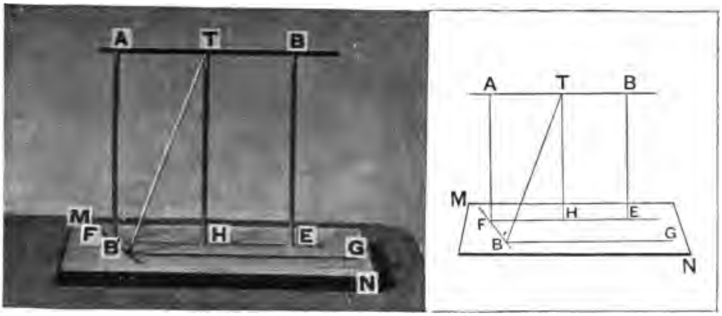
Therefore angle $SAC <$ angle SAB. § 93

Q. E. D.

586. *Def.*—The acute angle which a straight line makes with its own projection upon a plane is called the **inclination of the line to the plane.**

PROPOSITION XXIV. THEOREM

587. *Between two straight lines not in the same plane a common perpendicular can be drawn, and only one.*



GIVEN—AB and FB', two straight lines not in the same plane.

TO PROVE—that a common perpendicular can be drawn between them, and only one.

Through any point B' of FB' draw a line $B'G$ parallel to AB and let MN be the plane containing FB' and $B'G$.

MN is parallel to AB. § 550

Pass a plane through AB perpendicular to the plane MN, intersecting FB' at F and MN in FE.

FE is parallel to AB. § 546

At F erect a perpendicular FA to FE in the plane FB, hence perpendicular to MN and to FB'. § 575

Since FA is perpendicular to FE, it is perpendicular to AB. § 36

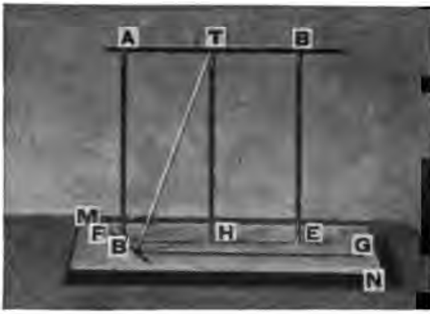
Therefore FA is a common perpendicular to AB and FB'.

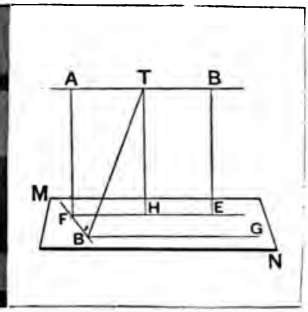
No other line as TB' can be perpendicular to both AB and FB'.

For TB' would also be perpendicular to $B'G$, parallel to AB.

Hence TB' would be perpendicular to MN. § 531

But TH drawn in AE perpendicular to FE is perpendicular to MN. § 575



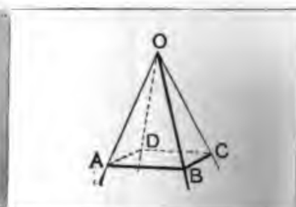


Hence there would be two perpendiculars from T to MN, which is impossible. § 537

Therefore TB' cannot be perpendicular to both AB and FB'. Q. E. D.

POLYEDRAL ANGLES

588. *Defs.*—When three or more planes meet in a point, they are said to form a **polyedral angle.**



Thus the planes AOB, BOC, COD, DOA passing through the common point O form the polyedral angle $O-ABCD$.

The common point O is called the **vertex** of the polyedral angle; the planes AOB, BOC, etc., are called the **faces**; the intersections OA, OB, etc., of the faces are called

the **edges**; the angles AOB, BOC, etc., are called the **face angles** of the polyedral angle.

The faces of a polyedral angle are supposed to be indefinite in extent. In order to show clearly in a figure the relative position of the edges, they are represented as being cut by a plane, as AC.

589. *Def.*—The polygon formed by the intersection of a plane with the faces of a polyedral angle is called a **section** of the polyedral angle.

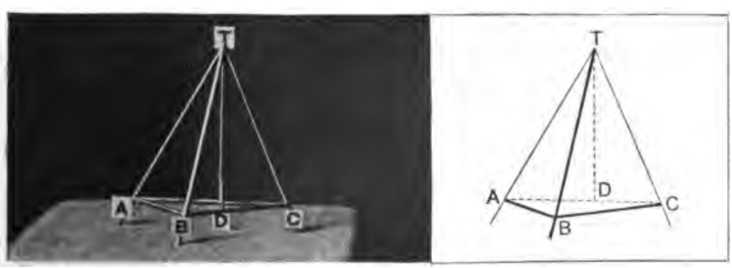
590. *Def.*—A polyedral angle is **convex** when any section by a plane forms a convex polygon.

591. *Def.*—The diedral angles formed by the faces, together with the face angles, are called the **parts** of the polyedral angle.

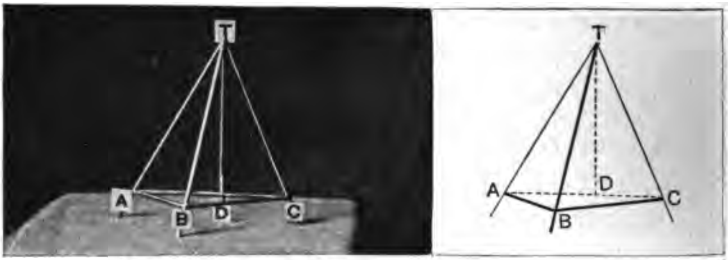
592. *Def.*—A polyedral angle of three faces is called a **triedral angle.**

PROPOSITION XXV. THEOREM

593. *The sum of any two face angles of a triedral angle is greater than the third.*



The theorem requires proof only when the third angle is greater than each of the others.



GIVEN—the triedral angle $T-ABC$ in which the face angle ATC is greater than either ATB or BTC.

TO PROVE $ATB+BTC>ATC$.

In the face ATC draw TD, making the angle $ATD=ATB$.

Take $TB=TD$, and through B and D pass a plane cutting the three faces in AB, BC, and AC.

The triangles ATB and ATD are equal. § 79

Hence $AB=AD$.

But $AB+BC>AC$.

By subtraction $BC>DC$.

The triangles BTC and DTC have two sides of one equal to two sides of the other, and the third side BC of one greater than the third side DC of the other.

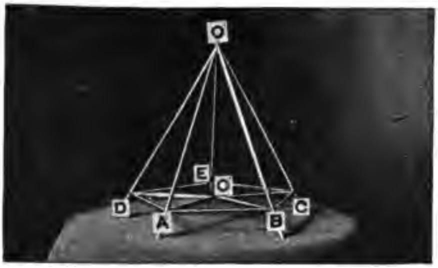
Therefore $BTC>DTC$. § 93

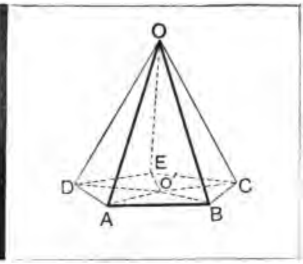
By construction $ATB=ATD$.

Adding $ATB+BTC>ATC$. Q. E. D.

PROPOSITION XXVI. THEOREM

594. *The sum of the face angles of any convex polyedral angle is less than four right angles.*





GIVEN the convex polyedral angle $O-ABCED$.

TO PROVE $AOB+BOC+$ etc. $<$ four right angles.

The section $ABCED$ of the polyedral angle is a convex polygon. § 590

Join any point O' in this polygon to its vertices.

In the triedral angle A we have

$$OAD+OAB>DAB. \qquad \text{§ 593}$$

Similarly $\quad OBA+OBC>ABC$, etc.

Adding these inequalities we get:

The sum of the base angles of the triangles about $O>$ the sum of the base angles of the triangles about O'.

But the sum of all the angles of the triangles about $O=$ the sum of all the angles of the triangles about O'.

[There being the same number of triangles having O for vertex as having O', and each triangle containing two right angles.]

Subtracting the inequality from the equality we get:

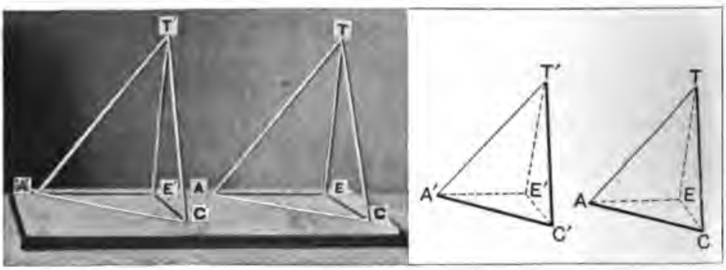
Sum of the angles whose vertex is $O<$ sum of the angles whose vertex is O'.

But the sum of the angles at O' is four right angles. § 28

Therefore the sum of the angles at O is less than four right angles. Q. E. D.

PROPOSITION XXVII. THEOREM

595. *Two triedral angles are equal, if two face angles and the included diedral angle of one are respectively equal to two face angles and the included diedral angle of the other, the parts given equal being arranged in the same order.*



GIVEN—the triedral angles $T-ACE$ and $T'-A'C'E'$ having angle CTA = angle $C'T'A'$; angle ETA = angle $E'T'A'$; diedral angle TA = diedral angle $T'A'$; the parts given equal being arranged in the same order.

TO PROVE $T-ACE = T'-A'C'E'$.

Place the triedral angles so that the equal diedral angles $T'A'$ and TA shall coincide, the point T' falling on T.

The angles $C'T'A'$ and CTA will then lie in the same plane.

Since they are equal, $T'C'$ will coincide with TC.

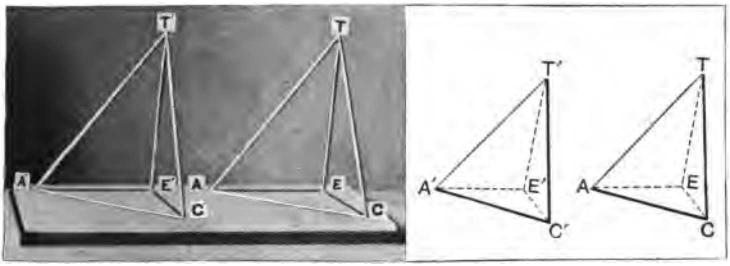
Similarly $T'E'$ will coincide with TE.

Then the third faces TEC and $T'E'C'$ will coincide. § 526 III

Therefore the triedral angles coincide throughout and are equal. Q. E. D.

PROPOSITION XXVIII. THEOREM

596. *Two triedral angles are equal, if two diedral angles and the included face angle of one are respectively equal to two diedral angles and the included face angle of the other, the parts given equal being arranged in the same order.*



GIVEN—the triedral angles $T-ACE$ and $T'-A'C'E'$ having face angle $CTA =$ face angle $C'T'A'$; diedral angle $TC =$ diedral angle $T'C'$; diedral angle $TA =$ diedral angle $T'A'$; the parts given equal being arranged in the same order.

TO PROVE $T-ACE = T'-A'C'E'$.

Place the triedral angles so that the equal angles $C'T'A'$ and CTA shall coincide.

Since diedral angle $T'C' =$ diedral angle TC, the plane TCE will take the direction of $T'C'E'$.

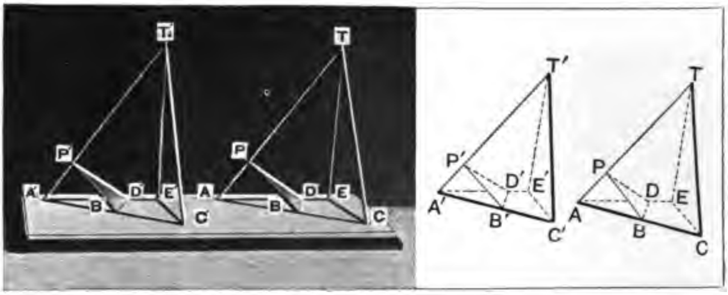
Similarly the plane TAE will take the direction of $T'A'E'$.

Then the intersection TE must lie somewhere in the plane $T'E'C'$ and somewhere in $T'E'A'$, and therefore must coincide with the intersection $T'E'$.

Therefore the triedral angles coincide throughout and are equal. Q. E. D.

PROPOSITION XXIX. THEOREM

597. *Two triedral angles are equal, if the three face angles of one are respectively equal to the three face angles of the other, provided the equal face angles are arranged in the same order.*



GIVEN—the triedral angles $T-ACE$ and $T'-A'C'E'$ having face angle $ATC=$face angle $A'T'C'$; face angle $CTE=$face angle $C'T'E'$; face angle $ETA=$face angle $E'T'A'$; the equal face angles being arranged in the same order.

TO PROVE $\quad T-ACE=T'-A'C'E'$.

On the six edges take $TA=TC=TE=T'A'=T'C'=T'E'$, and join AC, CE, EA, $A'C'$, $C'E'$, $E'A'$.

The triangles ATC and $A'T'C'$ are equal. § 79

Hence their homologous sides AC and $A'C'$ are equal. Similarly $CE=C'E'$ and $EA=E'A'$.

Therefore the triangles ACE and $A'C'E'$ are equal. § 89

At any point P in TA draw PB in the face ATC and PD in the face ATE perpendicular to TA.

PB must meet AC.

[For if PB were parallel to CA, CA would be perpendicular to TA (§ 36), which cannot be the case, since the angle TAC is acute, being a base angle of an isosceles triangle.]

And PB must meet AC upon that side of TA on which C lies.

[For if it met AC on the other side, there would be formed a triangle such that the sum of two of its angles, those at P and A, would be greater than two right angles, which is impossible.]

Likewise PD must meet EA on that side of TA on which E lies.

Let the points of meeting be B and D. Join BD.

On the edge $T'A'$ take $A'P'=AP$, and at P' repeat the same construction in the triedral angle T'.

The right triangles APB and $A'P'B'$ are equal. § 86

[Having a side, PA, and acute angle PAB of one equal to a side and homologous acute angle of the other.]

Therefore the homologous sides AB and $A'B'$ and PB and $P'B'$ are respectively equal.

Similarly $AD=A'D'$ and $PD=P'D'$.

Next, the triangles BAD and $B'A'D'$ are equal. § 79

[Having two sides AB and AD and the included angle DAB of one equal to two sides and the included angle of the other.]

Hence $BD=B'D'$.

Finally, the triangles PBD and $P'B'D'$ are equal. § 89

Therefore the homologous angles BPD and $B'P'D'$, that is, the plane angles of the diedral angles TA and $T'A'$, are equal.

Therefore the diedral angles AT and $A'T'$ are also equal. § 572

Therefore the triedral angles $T—ACE$ and $T'—A'C'E'$ are equal. § 595

Q. E. D.

598. *Outline of steps* used in the last proposition:

I. Proof that the *large face* triangles, viz., TAC and $T'A'C'$, etc., are equal.
II. Proof that the *large base* triangles, viz., ACE and $A'C'E'$, etc., are equal.
III. Proof that the *small face* triangles, viz., APB and $A'P'B'$, etc., are equal.
IV. Proof that the *small base* triangles, viz., ABD and $A'B'D'$, etc., are equal.
V. Proof that the triangles PBD and $P'B'D'$ are equal.

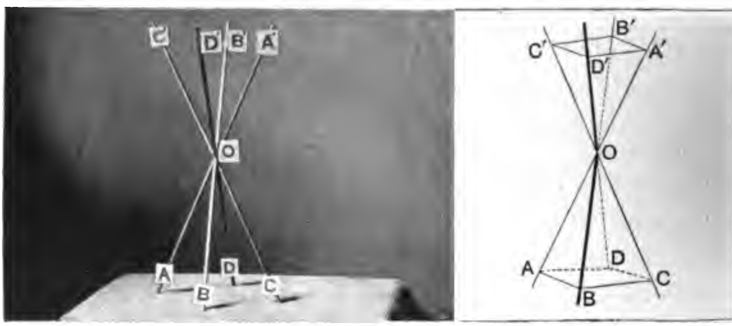
599. *Def.*—Two polyedral angles are **vertical,** if the edges of one are the prolongations, through the vertex, of the edges of the other.

600. *Def.*—Two polyedral angles are **symmetrical,** if all the parts of one are equal to those of the other but arranged in opposite order.

Symmetrical polyedral angles are not in general equal, that is, cannot be made to coincide, just as we cannot put a right glove on the left hand.

PROPOSITION XXX. THEOREM

601. *Two vertical polyedral angles are symmetrical.*



GIVEN—the vertical polyedral angles $O-ABCD$ and $O-A'B'C'D'$.

TO PROVE them symmetrical.

The lines OA', OB', etc., are the prolongations of the lines OA, OB, etc., respectively.

Therefore the angles $A'OB'$, $B'OC'$, etc., are equal respectively to the angles AOB, BOC, etc. § 30

The planes $A'OB'$, $B'OC'$, etc., are the prolongations of the planes AOB, BOC, etc., respectively. § 526 III

Hence the diedral angles OA', OB', etc., are equal respectively to the diedral angles OA, OB, etc. § 573

[Vertical diedral angles are equal.]

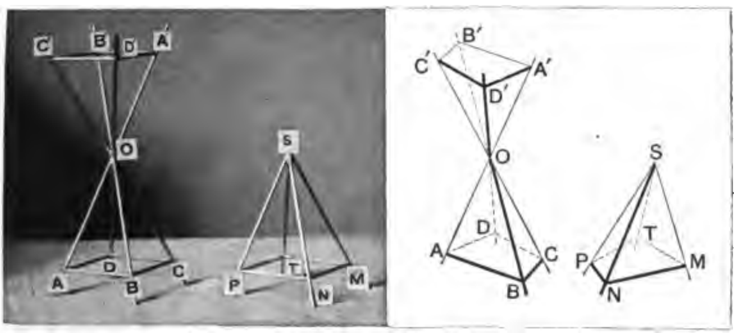
But the equal parts of the two polyedral angles are arranged in opposite order.*

Therefore they are symmetrical. § 600

[Having all the parts of one equal to those of the other, but arranged in opposite order.] Q. E. D.

PROPOSITION XXXI. THEOREM

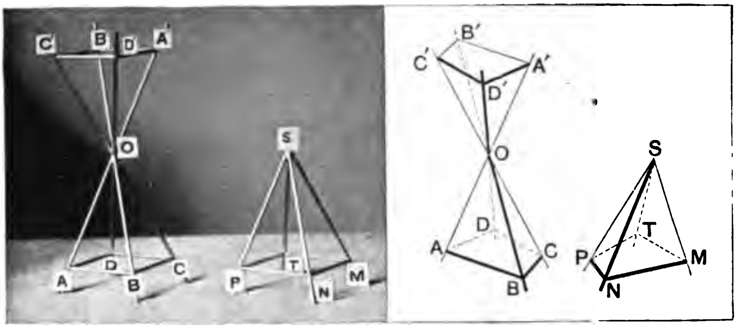
602. *Either of two symmetrical polyedral angles is equal to the vertical of the other.*



GIVEN—two symmetrical polyedral angles, $O-ABCD$ and $S-MNPT$, the points M, N, P, T, corresponding to the points A, B, C, D.

TO PROVE—that $S-MNPT$ can be made to coincide with $O-A'B'C'D'$, the vertical of $O-ABCD$.

* A convenient way of seeing this is to conceive the eye placed at O. Then, if we look at the points $A'B'C'D'$, we find that they follow each other in an order of rotation in the same direction as the hand of a clock moves. This order is called "clockwise." But if we look at $ABCD$, still keeping the eye at O, the order $ABCD$ is "counter-clockwise."



The parts of $S-MNPT$ and $O-ABCD$ are equal each to each and arranged in opposite order. § 600

[Two symmetrical polyedral angles have their parts equal each to each and arranged in opposite order.]

Also the parts of $O-A'B'C'D'$ and $O-ABCD$ are equal each to each and arranged in opposite order. § 601

[Two vertical polyedral angles are symmetrical.]

Therefore the parts of $S-MNPT$ and $O-A'B'C'D'$ are equal each to each and arranged in the same order.

Place the polyedral angle $S-MNPT$ so that its diedral angle SM shall coincide with the equal diedral angle OA', the point S falling at O.

Since the parts of the two polyedral angles are arranged in the same order, the angles NSM and $B'OA'$ will then lie in the same plane.

Since they are equal, SN will coincide with OB'.

Similarly we can show that the next edge SP will coincide with OC' and so on until all the edges and therefore all the faces coincide.

Hence the polyedral angles $S-MNPT$ and $O-A'B'C'D'$ coincide and are equal. Q. E. D.

PROPOSITION XXXII. THEOREM

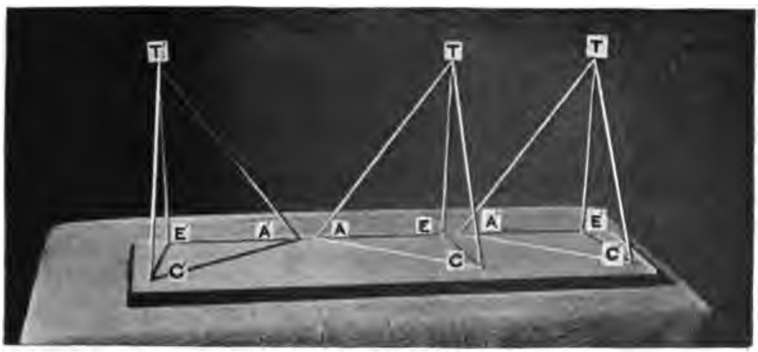
603. *Two triedral angles are symmetrical:*

I. *If two face angles and the included diedral angle of one are respectively equal to two face angles and the included diedral angle of the other.*

II. *If two diedral angles and the included face angle of one are respectively equal to two diedral angles and the included face angle of the other.*

III. *If the three face angles of one are respectively equal to the three face angles of the other.*

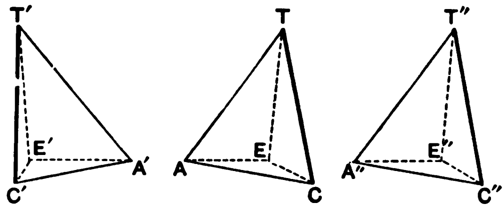
Provided the parts given equal are arranged in opposite order.



Let $T-ACE$ (or T) and $T'-A'C'E'$ (or T') be the two triedral angles, the parts given equal being arranged in opposite order.

Also let T'' be a triedral angle symmetrical to T', that is, having corresponding parts equal, but arranged in opposite order.

Therefore T'' and T have parts equal each to each and arranged in the same order.



Therefore in either of the three cases T is equal to T''. §§ 595, 596, 597

But T'' was constructed symmetrical to T'.

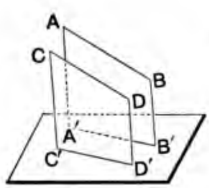
Therefore T, which equals T'', is also symmetrical to T'. Q. E. D.

604. *Def.*—A triedral angle is **isosceles,** if two of its face angles are equal.

605. *Exercise.*—If one of two symmetrical triedral angles is isosceles, the other is also, and the two can be made to coincide and are equal.

It will be noted, however, that the parts which correspond by symmetry will not be the ones which coincide.

PROJECTIONS



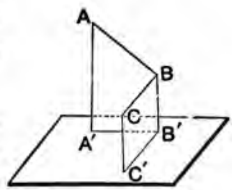
606. *Exercise.*—The projections on a plane of parallel lines are parallel.

Hint.—Prove first that the projecting planes are parallel, using § 551.

This principle is of great importance in the theory of shades and shadows.

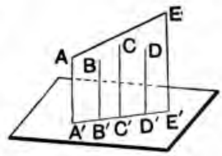
It is not true in general that if two lines make an angle with each other, their projections on a plane will make the same angle.

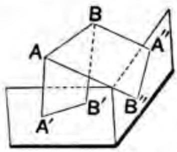
607. *Exercise.*—The projection on a plane of a right angle is a right angle provided one of the sides is parallel to the plane.



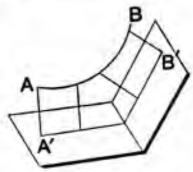
Hint.—Prove first that the side which is parallel to the plane is perpendicular to the projecting plane of the other; then that the two projecting planes are perpendicular, and, finally, that the projections of the sides are perpendicular.

608. *Exercise.*—If the projections on a plane of a number of points lie in a straight line, the points must lie in a plane.



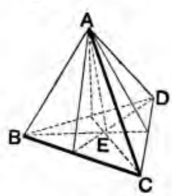


609. *Exercise.*—If the projections of a line on each of two intersecting planes be straight, the line itself must be straight except in one case. State that case.

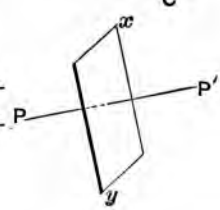


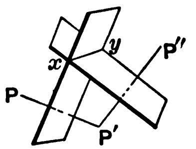
LOCI

610. *Exercise.*—In any triedral angle the three planes bisecting the three diedral angles intersect in a common straight line, which is the locus of points within the triedral angle equidistant from its faces.

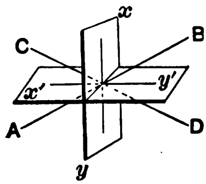


611. *Exercise.*—Find, and prove correct, the locus of all points in space equidistant from two given points.





612. *Exercise.*—Find, and prove correct, the locus of all points equidistant from three given points.



613. *Exercise.*—The locus of points equidistant from two intersecting straight lines is the pair of planes passed through the bisectors of the angles formed by the lines and perpendicular to the plane of the lines.

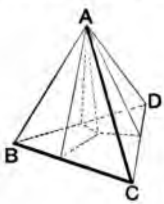
Hint.—Apply §§ 595, 86.

614. *Exercise.*—Find the locus of points equidistant from a given plane.

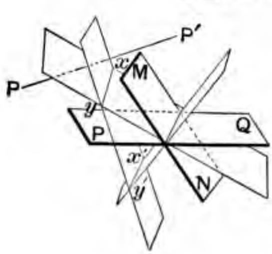
615. *Exercise.*—Find the locus of points equidistant from two parallel planes.

616. *Exercise.*—Find the locus of points equidistant from two intersecting planes.

617. *Exercise.*—Find the locus of points equidistant from three intersecting straight lines not in the same plane.

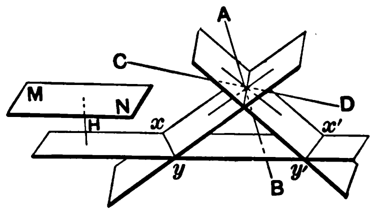


618. *Exercise.*—In any triedral angle the three planes passed through the bisectors of the three face angles, and perpendicular to these faces respectively, intersect in a common straight line, every point of which is equidistant from the edges of the triedral angle.



619. *Exercise.*—Find, and prove correct, the locus of points which are equidistant from two given planes, and at the same time equidistant from two given points.

620. *Exercise.* — Find, and prove correct, the locus of points at a given distance from a given plane, and at the same time equidistant from two intersecting straight lines.



Does the figure show all the lines of the locus?

PROBLEMS OF CONSTRUCTION

The constructions of solid geometry differ from those of plane geometry in that we cannot perform them with ruler and compasses, or with any instruments of drawing.

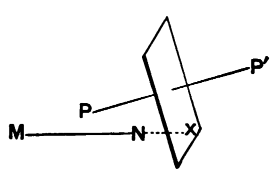
We shall therefore consider a problem of construction in solid geometry solved when it is reduced to one or more of the following elementary constructions which we assume can be performed, viz.:

(1.) A plane can be drawn through any three given points.

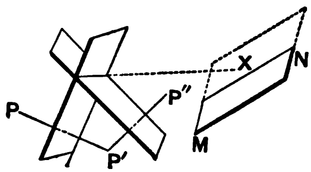
(2.) The intersection of a plane with any given straight line or with any given plane can be determined.

(3.) A straight line can be drawn through any given point perpendicular to any given plane.

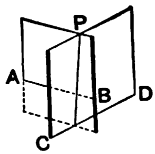
621. *Exercise.* — Determine a point in a given straight line which shall be equidistant from two given points in space.



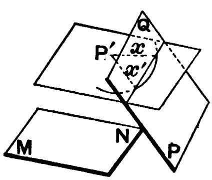
Do not assume that the given line and the given points are in the same plane, and avoid similar assumptions in the following exercises.



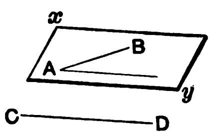
622. *Exercise.*—Determine a point in a plane MN which shall be equidistant from three given points in space, P, P', and P''.



623. *Exercise.*—Through a given point P in space determine a straight line which shall cut two given straight lines AB and CD.

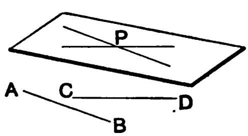


624. *Exercise.*—Given a point P' and any two non-parallel planes MN and PQ. From the point draw a straight line of given length terminating in one of the planes and parallel to the other.



625. *Exercise.*—Show how to pass a plane through a straight line AB parallel to another straight line CD.

Hint.—Apply § 550.



626. *Exercise.*—Show how to pass a plane through a point P parallel to two given straight lines AB and CD.

GEOI

627. *Defs.*—A
by planes.

The intersection
edges of the polye
called the **vertice**
bounded by the ed

The least number of
by intersecting form a
close with these a defi

628. *Defs.*—A
dron; one of six
an **octaedron**; one
twenty faces, an **ic**

ICOSAEDRON DODECA

629. *Def.*—A polyedron is **convex** when no face, if produced, will enter the polyedron.

All the polyedrons treated of in this book will be understood to be convex.

PRISMS. PARALLELOPIPEDS

630. *Defs.*—A **prismatic surface** is a surface composed of planes passed between each successive pair of a system of parallel lines.

The parallel lines are called the **edges** of the prismatic surface.

PROPOSITION I. THEOREM

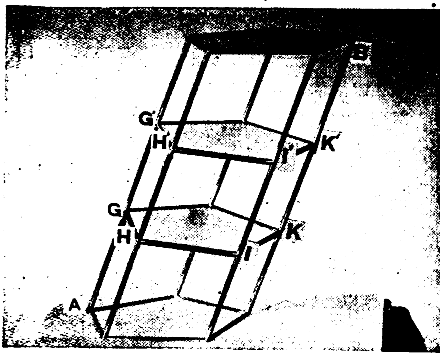
631. *The sections of a prismatic surface made by two parallel planes cutting its edges are equal polygons.*

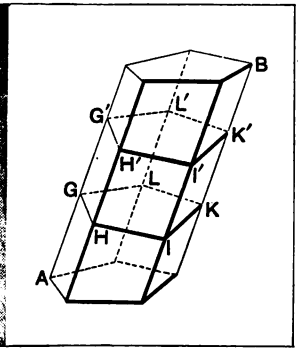
GIVEN—the prismatic surface AB cut by two parallel planes in the sections $GHIKL$ and $G'H'I'K'L'$.

TO PROVE these polygons are equal.

The sides GH, HI, etc., are parallel respectively to $G'H'$, $H'I'$, etc. § 544

Hence $GH=G'H'$, $HI=H'I'$. § 118





Also angle GHI = angle $G'H'I'$,

angle HIK = angle $H'I'K'$, etc. § 557

The polygons $GHIKL$ and $G'H'I'K'L'$ are therefore mutually equilateral and equiangular.

Hence they can be made to coincide and are equal. Q. E. D.

632. COR. *A prismatic surface can be generated by a straight line moving so as to remain always parallel to a fixed straight line (drawn parallel to the edges) and always cutting the perimeter of a section.*

Hint.—By plane geometry a straight line can move across each face remaining parallel to the lateral edges.

633. *Defs.*—A **prism** is a polyedron bounded by a prismatic surface and two parallel planes.

PRISMS

The equal sections of the prismatic surface formed by the parallel planes are called the **bases** of the prism; the portion of the prismatic surface between the bases consists of the **lateral faces**; the portions of the edges of the prismatic surface between the bases are the **lateral edges** of the prism.

634. *Defs.*—A **right prism** is one whose lateral edges are perpendicular to its bases.

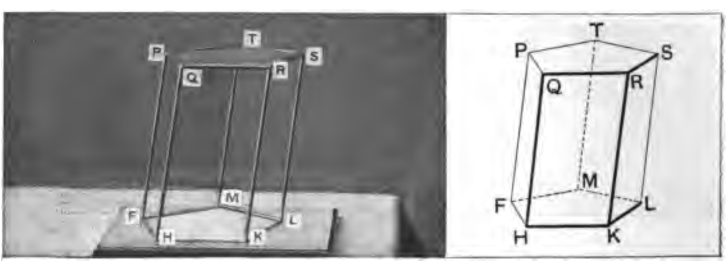
An **oblique prism** is one whose lateral edges are not perpendicular to its bases.

635. *Def.*—A **regular prism** is one whose bases are regular polygons and whose lateral edges are perpendicular to its bases.

RIGHT PRISM REGULAR PRISM OBLIQUE PRISMS

PROPOSITION II. THEOREM

636. *The lateral faces of a prism are parallelograms.*



GIVEN the prism FS.

TO PROVE its lateral faces are parallelograms.

Consider the lateral face FQ.

Its sides FP and HQ are parallel, being edges of the prismatic surface. § 630

Also FH and PQ are parallel, being the intersections of two parallel planes with a third. § 544

Therefore FQ is a parallelogram. § 114

Similarly the other lateral faces are proved to be parallelograms. Q. E. D.

637. COR. I. *The lateral edges of a prism are equal.*

638. COR. II. *The lateral faces of a right prism are rectangles.*

639. *Def.*—A **parallelopiped** is a prism whose bases are parallelograms.

640. *Def.*—A **right parallelopiped** is a parallelopiped whose lateral edges are perpendicular to its bases.

OBLIQUE PARALLELOPIPED — RIGHT PARALLELOPIPED — RECTANGULAR PARALLELOPIPED — CUBE

641. *Def.*—A **rectangular parallelopiped** is a right parallelopiped whose bases are rectangles.

642. *Def.*—A **cube** is a right parallelopiped whose bases are squares and whose lateral edges are equal to the sides of its base.

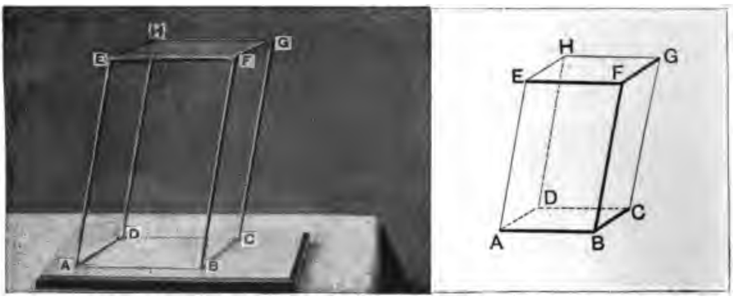
643. COR. III. *All the faces of a parallelopiped are parallelograms.*

644. COR. IV. *All the faces of a rectangular parallelopiped are rectangles.*

645. COR. V. *All the faces of a cube are equal squares.*

PROPOSITION III. THEOREM

646. *Any two opposite faces of a parallelopiped may be taken as its bases.*



GIVEN—the parallelopiped AG, the bases being first taken as AC and EG.

TO PROVE—that any other two opposite faces, as AF and DG, may be taken as bases.

The four lines AD, BC, FG, EH are parallel to each other. §§ 114, 549

They may therefore be taken as the edges of a prismatic surface. § 630

Also AB and AE are parallel to DC and DH respectively. § 114

Hence the planes AF and DG are parallel. § 551

Therefore the parallelopiped may be considered a prism having AF and DG as bases. § 633

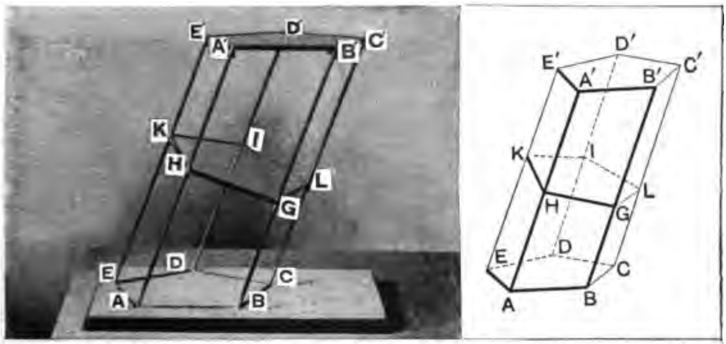
Q. E. D.

647. *Def.*—A **right section** of a prism is the section formed by a plane perpendicular to the lateral edges.

648. *Def.*—The **lateral area** of a prism is the sum of the areas of its lateral faces.

PROPOSITION IV. THEOREM

649. *The lateral area of a prism is equal to the product of the perimeter of a right section and a lateral edge.*



GIVEN—the prism AC', of which $HGLIK$ is a right section.

TO PROVE—its lateral area $= (HG + GL + \text{etc.}) \times AA'$.

The lateral area consists of the areas of the lateral faces, which are parallelograms. § 636

The area of each parallelogram is its base multiplied by its altitude. § 385

Their bases AA', BB', etc., are all equal. § 637

Their altitudes are the lines HG, GL, etc. § 530

Hence by addition we have

$$\text{lateral area} = (HG + GL + \text{etc.}) \times AA'. \qquad \text{Q. E. D.}$$

650. *Def.*—The **altitude** of a prism is the perpendicular distance between the planes of its bases.

651. COR. *The lateral area of a right prism is equal to the product of the perimeter of its base and its altitude.*

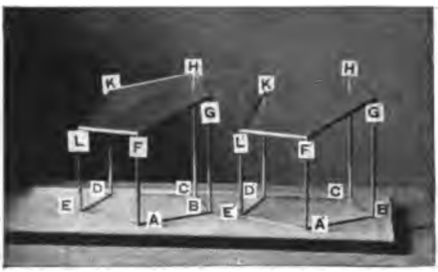
652. *Defs.*—A **truncated prism** is a polyedron bounded by a prismatic surface and two non-parallel planes.

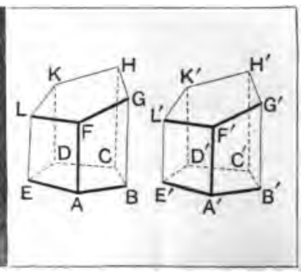
The sections of the prismatic surface formed by the non-parallel planes are called the **bases** of the truncated prism.

653. *Def.*—A truncated prism is **right** when one of its bases is perpendicular to the lateral edges.

PROPOSITION V. THEOREM

654. *Two right truncated prisms are equal, if three lateral edges of one are equal to three corresponding edges of the other and the bases to which they are respectively perpendicular are equal.*





GIVEN—the truncated right prisms AK and $A'K'$, having the lateral edges AF and $A'F'$, BG and $B'G'$, CH and $C'H'$ respectively equal and perpendicular to the equal bases $ABCDE$, $A'B'C'D'E'$.

TO PROVE that AK and $A'K'$ are equal.

Superpose the truncated prisms so that the bases $ABCDE$ and $A'B'C'D'E'$ shall coincide.

Then the indefinite lines AF, BG, etc., will coincide respectively with $A'F'$, $B'G'$, etc. § 535

Hence the indefinite prismatic surfaces coincide. § 526 IV

Since $AF=A'F'$, F falls on F'. Similarly G falls on G' and H upon H'.

Hence the planes of the upper bases coincide. § 526 I

Therefore the truncated prisms coincide and are equal.

Q. E. D.

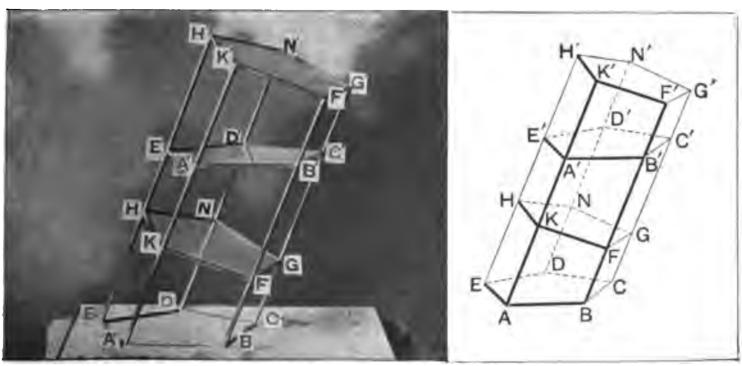
655. COR. *Two right prisms are equal, if they have equal bases and equal altitudes.*

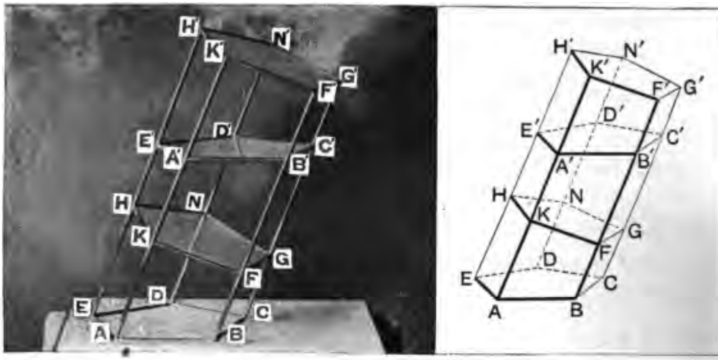
656. *Defs.*—The **volume** of any solid is its ratio to another solid taken arbitrarily and called the **unit of volume.**

657. *Def.*—Two solids are **equivalent** when their volumes are equal.

PROPOSITION VI. THEOREM

658. *An oblique prism is equivalent to a right prism whose base is a right section of the oblique prism and whose altitude is equal to a lateral edge of the oblique prism.*





GIVEN—the oblique prism *ABCDE-A'* of which *KFGNH* is a right section.

Produce *AA'* to *K'*, making $KK' = AA'$, and through *K'* pass a plane parallel to *KFGNH*, cutting the other edges produced in *F' G', N', H'*.

TO PROVE—the oblique prism *ABCDE-A'* is equivalent to the right prism *KFGNH-K'*.

The truncated right prisms *AG* and *A'G'* have the bases *KFGNH* and *K'F'G'N'H'* equal. § 631

Also the lateral edges *AK*, *BF*, and *CG* are respectively equal to *A'K'*, *B'F'*, and *C'G'*. Ax. 3

Therefore these truncated prisms are equal. § 654

If we take the upper truncated prism *A'G'* from the whole figure, we have left the oblique prism.

If we take the lower truncated prism *AG* from the whole figure, we have left the right prism.

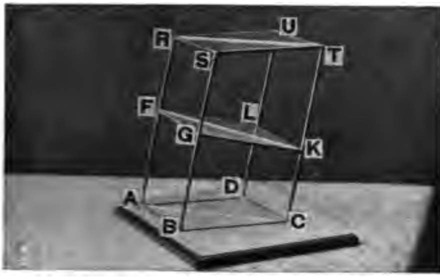
Therefore the oblique prism is equivalent to the right prism. Ax. 3

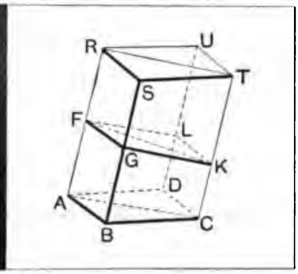
Q. E. D.

659. *Defs.*—A **triangular prism** is one whose base is a triangle; a **quadrangular**, one whose base is a quadrilateral.

PROPOSITION VII. THEOREM

660. *The plane passed through two diagonally opposite edges of a parallelopiped divides it into two equivalent triangular prisms.*





GIVEN—the parallelopiped $ABCD$-R divided by the plane $ARTC$ into two triangular prisms ABC-R and ACD-U.

TO PROVE these triangular prisms are equivalent.

Let $FGKL$ be a right section of the parallelopiped, cutting the plane $ARTC$ in FK.

The planes AU and BT are parallel. § 551

Therefore FL and GK are parallel. § 544

Similarly FG and LK are parallel.

Therefore $FGKL$ is a parallelogram. § 114

Hence the triangles FGK and FKL are equal. § 116

Now the triangular prism ABC-R is equivalent to a right prism whose base is the right section FGK and whose altitude is the lateral edge AR, and ACD-U is equivalent to a right prism whose base is FKL and whose altitude is AR. § 658

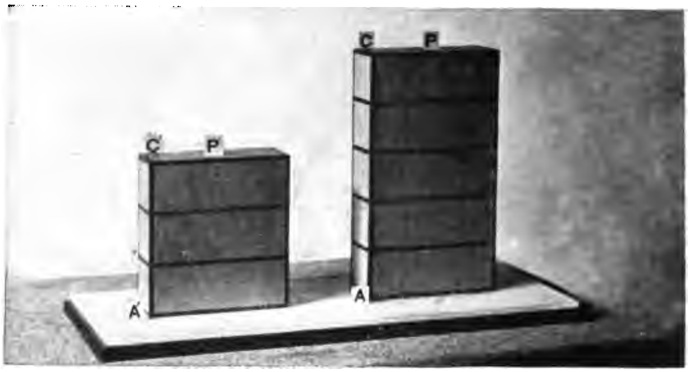
These two right prisms are equal. § 655

Therefore ABC-R and ACD-U are equivalent. Ax. 1

Q. E. D.

PROPOSITION VIII. THEOREM

661. *Two rectangular parallelopipeds having equal bases are to each other as their altitudes.*



GIVEN—the rectangular parallelopipeds P and P' having equal bases, their altitudes being AC and $A'C'$.

TO PROVE $$\frac{P'}{P}=\frac{A'C'}{AC}.$$

CASE I. *When the altitudes are commensurable.*

Suppose the common measure of AC and $A'C'$ to be contained in AC 5 times, and in $A'C'$ 3 times.

Then $$\frac{A'C'}{AC}=\frac{3}{5}.$$

Through the points of division of AC and $A'C'$ pass planes parallel to the bases.

These planes divide the parallelopipeds into smaller parallelopipeds, all of which are equal. §§ 631, 655

P contains 5 and P' contains 3 of these small parallelopipeds.

Hence $$\frac{P'}{P}=\frac{3}{5}.$$

Therefore $$\frac{P'}{P}=\frac{A'C'}{AC}.$$ Ax. 1

CASE II. *When the altitudes are incommensurable.*

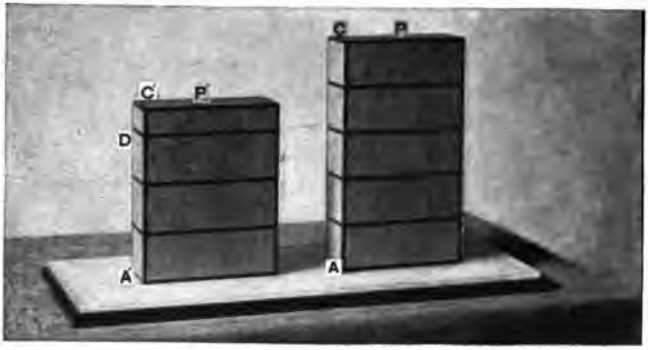
Divide AC into any number of equal parts and apply one of these parts to $A'C'$ as often as $A'C'$ will contain it.

Since AC and $A'C'$ are incommensurable, there will be a remainder DC' less than one of these parts.

Pass a plane through D parallel to the bases of P' and let X be the rectangular parallelopiped between this plane and the *lower* base of P'.

Then, since $A'D$ and AC are *commensurable*, we have

$$\frac{X}{P}=\frac{A'D}{AC}.$$ Case I



If each of the parts of AC be continually bisected, each part can be made as small as we please.

Therefore DC', which is always less than one of these parts, can be made as small as we please.

But it can never be reduced to zero, since AC and $A'C'$ are given incommensurable.

Therefore $A'D$ will approach $A'C'$ as a limit. § 185

Hence $\frac{A'D}{AC}$ will approach $\frac{A'C'}{AC}$ as a limit. § 190

Likewise $\frac{X}{P}$ will approach $\frac{P'}{P}$ as a limit.

Therefore $$\frac{P'}{P}=\frac{A'C'}{AC}.$$ § 186

Q. E. D.

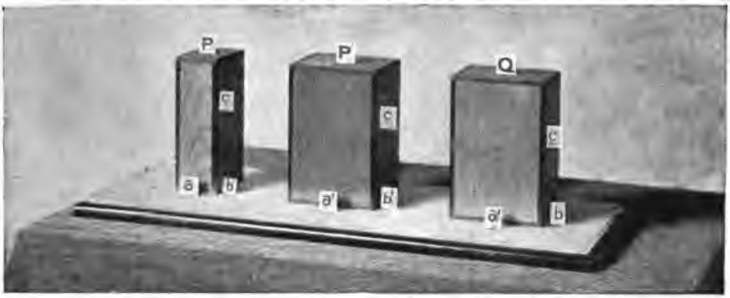
662. *Def.*—The three edges of a rectangular parallelopiped meeting at a common vertex are called its **dimensions.**

663. *Remark.*—The preceding theorem may be stated thus:

Two rectangular parallelopipeds which have two dimensions in common are to each other as their third dimensions.

PROPOSITION IX. THEOREM

664. *Two rectangular parallelopipeds which have one dimension in common are to each other as the products of the two other dimensions.*



GIVEN—the rectangular parallelopipeds P and P', having the dimension c common, the other dimensions being a, b and a', b' respectively.

TO PROVE $$\frac{P}{P'}=\frac{a\times b}{a'\times b'}.$$

Let Q be a third rectangular parallelopiped having the dimensions a', b, c.

Then P and Q have two dimensions b and c in common.

Hence $$\frac{P}{Q}=\frac{a}{a'}. \qquad \S\ 663$$

Also Q and P' have two dimensions a' and c in common.

Hence $$\frac{Q}{P'}=\frac{b}{b'}.$$

Multiplying these equations together, we get

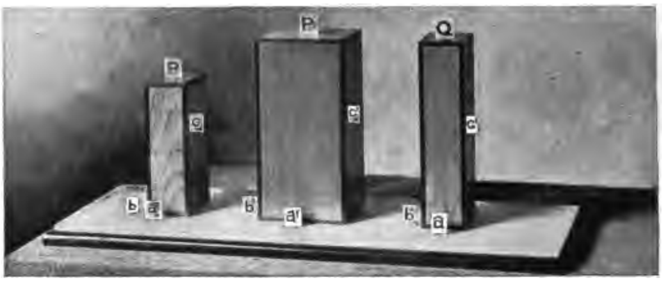
$$\frac{P}{P'}=\frac{a\times b}{a'\times b'}.$$ Q. E. D.

665. *Remark.*—This theorem may be stated thus:

Two rectangular parallelopipeds having equal altitudes are to each other as their bases.

PROPOSITION X. THEOREM

666. *Any two rectangular parallelopipeds are to each other as the products of their three dimensions.*



GIVEN—the rectangular parallelopipeds P and P', whose dimensions are a, b, c and a', b', c' respectively.

TO PROVE $$\frac{P}{P'}=\frac{a\times b\times c}{a'\times b'\times c'}.$$

Let Q be a third rectangular parallelopiped having the dimensions a, b, c'.

Then $$\frac{P}{Q}=\frac{c}{c'}.$$ § 663

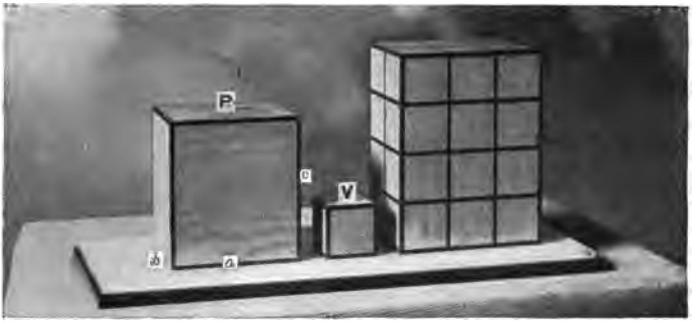
And $$\frac{Q}{P'}=\frac{a\times b}{a'\times b'}.$$ § 664

Multiplying these equations together, we have

$$\frac{P}{P'}=\frac{a\times b\times c}{a'\times b'\times c'}.$$ Q. E. D.

PROPOSITION XI. THEOREM

667. *The volume of a rectangular parallelopiped is equal to the product of its three dimensions, provided that the unit of volume is a cube whose edge is the linear unit.*



Proof.—Let P be any rectangular parallelopiped whose dimensions are a, b, and c, and let the cube V, whose edge is the linear unit, be the unit of volume.

Then $\frac{P}{V}$ is the volume of P. § 656

But $\frac{P}{V}=\frac{a\times b\times c}{1\times 1\times 1}=a\times b\times c.$ § 666

Therefore vol. $P=a\times b\times c.$ Q. E. D.

668. *Remark.*—Hereafter the unit of volume is to be understood to be a cube whose edge is the linear unit.

669. *Remark.*—This theorem may also be stated:

The volume of a rectangular parallelopiped is equal to the product of its base and altitude.

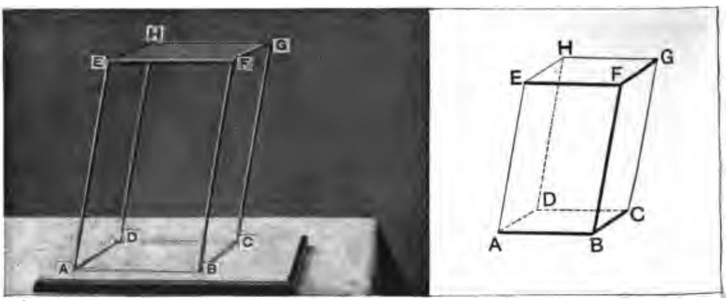
670. COR. II. *The volume of a cube is equal to the third power of its edge.*

Hence it is that the *third power* of a number is called the *cube* of that number.

671. *Remark.*—When the three dimensions of a rectangular parallelopiped are exactly divisible by the linear unit, the truth of the proposition may be rendered evident by dividing the parallelopiped into cubes, whose edges are equal to the linear unit.

Thus, if three edges which meet at a common vertex are respectively 2 units, 3 units, and 4 units in length, the parallelopiped may be divided into 24 cubes, each equal to the unit of volume, by passing planes perpendicular to the edges through their points of division.

672. CONSTRUCTION. *To construct a parallelopiped having as edges three given straight lines drawn from the same point.*



GIVEN the straight lines AB, AD, and AE.

TO CONSTRUCT—a parallelopiped having them as edges.

Pass a plane through each pair of the given straight lines.

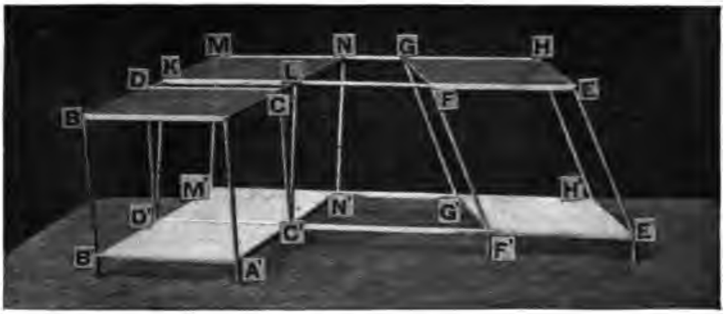
Then pass a plane through the extremity of each line parallel to the plane of the other two.

The solid thus formed may be shown to be a parallelopiped by applying successively §§ 544, 630, 633, 639.

673. *Exercise.*—Show that if the three given lines in the preceding construction are perpendicular to each other, the parallelopiped formed will be rectangular.

PROPOSITION XII. THEOREM

674. *The volume of any parallelopiped is equal to the product of its base and altitude.*



GIVEN—the *oblique* parallelopiped FH', whose base is $F'E'H'G'$ and altitude h.

TO PROVE vol. $FH' = F'E'H'G' \times h$.

Produce the edges EF, HG, $E'F'$, $H'G'$, and in $E'F'$ produced take $C'D' = E'F'$.

Through C' and D' pass planes perpendicular to $E'D'$, forming the *right* parallelopiped KN'.

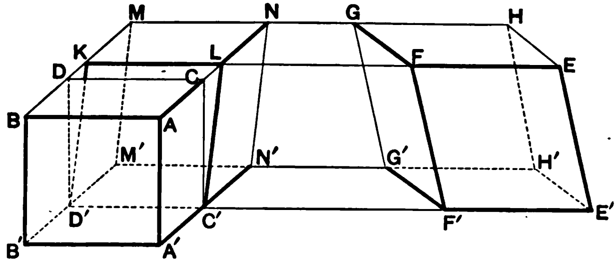
Now produce the edges $N'C'$, NL, MK, $M'D'$ of the right parallelopiped and in $N'C'$ produced take $C'A' = N'C'$.

In the plane $A'N$ draw $C'C$ perpendicular to $A'C'$.

The three lines $C'D'$, $C'A'$, $C'C$ are perpendicular to each other. § 530

Therefore the parallelopiped BC' formed upon them as edges will be rectangular. § 673

The rectangular and oblique parallelopipeds are each equivalent to the right parallelopiped, and therefore to each other. § 658



Their bases $F'E'H'G'$ and $B'A'C'D'$ are each equivalent to $D'C'N'M'$, and therefore to each other. § 386

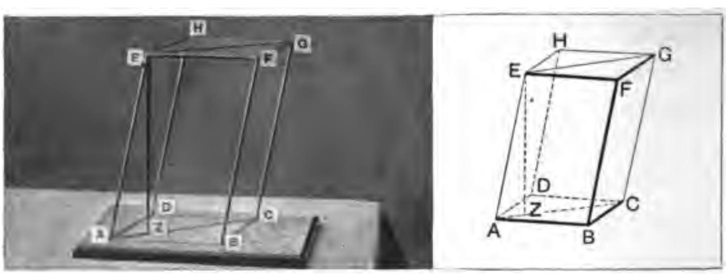
And the altitude of each is h. § 565

But vol. $BC' = B'A'C'D' \times h$. § 667

Therefore vol. $FH' = F'E'H'G' \times h$. Q. E. D.

PROPOSITION XIII. THEOREM

675. *The volume of a triangular prism is equal to the product of its base and altitude.*



GIVEN—the triangular prism $ABC\text{-}F$ having the base ABC and altitude EZ.

TO PROVE vol. $ABC\text{-}F = ABC \times EZ$.

Construct the parallelopiped $ABCD\text{-}F$ having AB, AD, and AE as edges. § 672

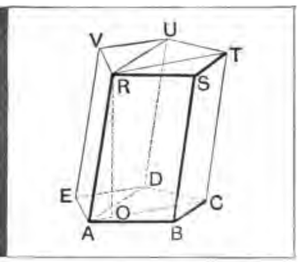
Then the volume of the parallelopiped is the product of its base $ABCD$ and its altitude EZ. § 674

But the volume of the triangular prism is half the volume of the parallelopiped; its base is half the base of the parallelopiped; and its altitude is the same. §§ 660, 116

Therefore the volume of the triangular prism is the product of its base ABC and its altitude EZ. Q. E. D.

PROPOSITION XIV. THEOREM

676. *The volume of any prism is equal to the product of its base and altitude.*



GIVEN—the prism $ABCDE\text{-}R$ with base $ABCDE$ and altitude RO.

TO PROVE vol. $ABCDE\text{-}R = ABCDE \times RO$.

The prism may be divided into triangular prisms by planes passed through AR and the diagonally opposite edges.

The volume of each triangular prism is the product of its base and altitude. § 675

They have the common altitude RO. § 565

Therefore the volume of the whole prism is the sum of the bases of the triangular prisms, i. e., the base of the whole prism, multiplied by the common altitude. Q. E. D.

677. COR. I. *Two prisms having equivalent bases and equal altitudes are equivalent.*

678. COR. II. *Any two prisms are to each other as the products of their bases and altitudes.*

Hint.—Prove as in § 387.

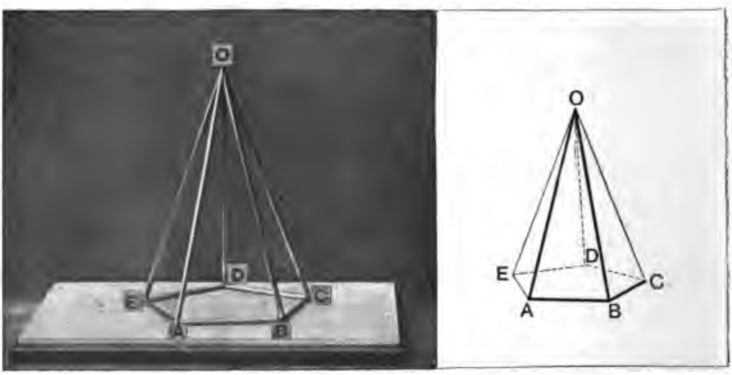
679. COR. III. *Two prisms having equivalent bases are to each other as their altitudes.*

680. COR. IV. *Two prisms having equal altitudes are to each other as their bases.*

PYRAMIDS

681. *Defs.*—A **pyramid** is a polyedron one of whose faces is a polygon and whose other faces are triangles having the sides of the polygon for bases and a common vertex outside the plane of the polygon.

The polygon is the **base**; the triangles are the **lateral faces**; the common vertex of the triangles is the **vertex** of the pyramid; and the edges passing through the vertex are its **lateral edges**.

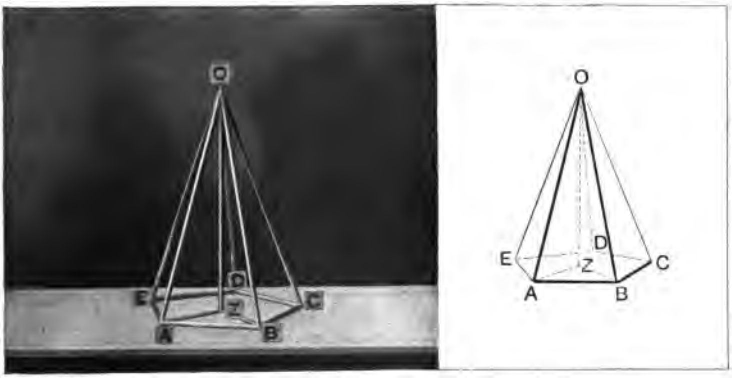


Thus $ABCDE$ is the base; O is the vertex; OA, OB, etc., are the lateral edges; and OAB, OBC, etc., are the lateral faces of the prism O-$ABCDE$.

682. *Defs.*—A **regular pyramid** is a pyramid whose base is a regular polygon and whose vertex lies in the perpendicular to the base erected at its centre. This perpendicular is called the **axis** of the regular pyramid.

PROPOSITION XV. THEOREM

683. *The lateral edges of a regular pyramid are equal.*



GIVEN the regular pyramid $O\text{-}ABCDE$.

TO PROVE $OA = OB = OC =$ etc.

Let OZ be the axis of the regular pyramid.

Then $ZA = ZB = ZC =$ etc. § 461 III

Therefore $OA = OB = OC =$ etc. § 539 I

Q. E. D.

684. COR. I. *The lateral faces of a regular pyramid are equal isosceles triangles.*

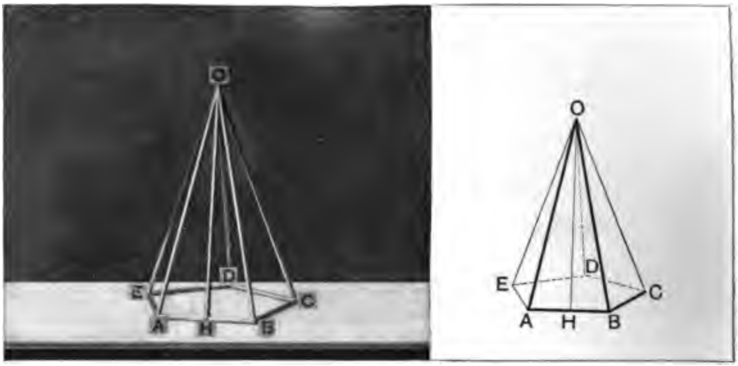
685. COR. II. *The altitudes of the lateral faces drawn from the common vertex O are equal.*

686. *Def.*—The **slant height** of a regular pyramid is the altitude of any one of its lateral faces drawn from the vertex of the pyramid.

687. *Def.*—The **lateral area** of a pyramid is the sum of the areas of its lateral faces.

PROPOSITION XVI. THEOREM

688. *The lateral area of a regular pyramid is equal to one-half the product of the perimeter of its base and its slant height.*



GIVEN—the regular pyramid $O\text{-}ABCDE$, of which OH is the slant height.

TO PROVE—lat. area $O\text{-}ABCDE = \frac{1}{2}(AB + BC + \text{etc.}) \times OH$.

The lateral area of the pyramid is composed of the areas of the triangles OAB, OBC, etc. § 687

The area of each triangle is half the product of its base and altitude.

Hence area $OAB = \frac{1}{2} AB \times OH$,

area $OBC = \frac{1}{2} BC \times OH$, etc. § 685

Therefore the lateral area of the pyramid is

$\frac{1}{2}(AB + BC + \text{etc.}) \times OH$. Q. E. D.

689. *Defs.*—A **truncated pyramid** is the portion of a pyramid contained between its base and a plane cutting all its lateral edges.

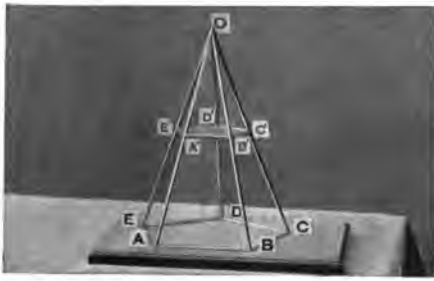
The section thus made, together with the base of the pyramid, are called the **bases** of the truncated pyramid.

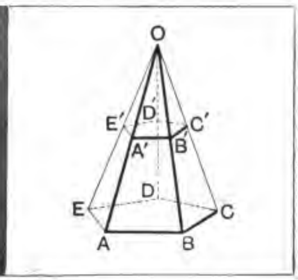
The other faces are the **lateral faces** of the truncated pyramid.

690. *Def.*—A **frustum of a pyramid** is a truncated pyramid, the planes of whose bases are parallel.

PROPOSITION XVII. THEOREM

691. *The lateral faces of a frustum of a regular pyramid are equal trapezoids.*





GIVEN—the frustum EC' of the regular pyramid $O\text{-}ABCDE$.

TO PROVE—its faces are equal trapezoids, viz.: $ABB'A'$, $BCC'B'$, etc.

The faces *are* trapezoids, since $A'B'$, $B'C'$, etc., are parallel to AB, BC, etc., respectively. § 544

Superpose the equal isosceles triangles OAB, OBC by turning the first over OB on to the second.

Then also must $A'B'$ coincide with $B'C'$, both being parallel to BC. Ax. *b*

Thus the two trapezoids coincide and are equal. Likewise all the trapezoids are equal. Q. E. D.

692. *Def.*—The **slant height** of a frustum of a regular pyramid is the altitude of any lateral face.

693. Cor. *The lateral area of a frustum of a regular pyramid equals one-half the product of the sum of the perimeters of its bases and its slant height.*

Hint.—Apply § 396.

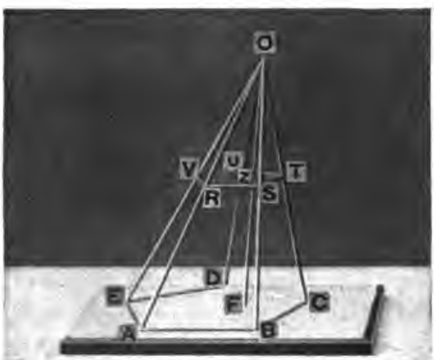
694. *Def.*—The **altitude** of a pyramid is the perpendicular distance from the vertex to the plane of the base.

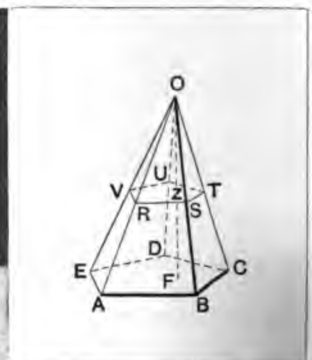
PROPOSITION XVIII. THEOREM

695. *If a pyramid is cut by a plane parallel to its base:*

I. *The lateral edges and the altitude are divided proportionally.*

II. *The section is a polygon similar to the base.*





Given—the pyramid $O—ABCDE$ cut by a plane parallel to its base $ABCDE$ in the section $RSTUV$, and the altitude OF cutting the plane of the section in Z.

I. To prove $\frac{OR}{OA}=\frac{OS}{OB}=\frac{OT}{OC}=$ etc. $=\frac{OZ}{OF}$.

This follows immediately from § 556.

II. To prove $RSTUV$ is similar to $ABCDE$.

The corresponding sides of the two polygons are parallel. § 544

Hence their angles are equal. § 557

Also the triangles ORS, OST, etc., are similar to OAB, OBC, etc. § 275

Hence $$\frac{RS}{AB}=\frac{OS}{OB}=\frac{ST}{BC}=\text{etc.}$$

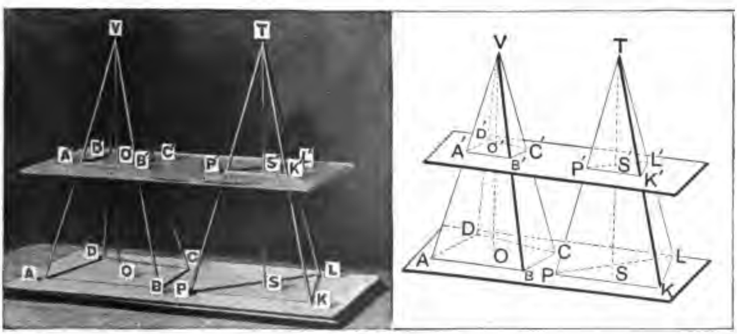
Hence $$\frac{RS}{AB}=\frac{ST}{BC}=\text{etc.}$$

Therefore $RSTUV$ is similar to $ABCDE$. Q. E. D.

696. Cor. I. *The areas of any sections of a pyramid parallel to its base are proportional to the squares of their distances from the vertex.*

Outline proof: $$\frac{\text{area } RSTUV}{\text{area } ABCDE}=\frac{\overline{RS}^2}{\overline{AB}^2}=\frac{\overline{OS}^2}{\overline{OB}^2}=\frac{\overline{OZ}^2}{\overline{OF}^2}.$$

697. Cor. II. *If two pyramids V-ABCD and T-PKL, having equal altitudes VO and TS, are cut by planes parallel to their bases at equal distances VO′ and TS′ from their vertices, the sections A′B′C′D′ and P′K′L′ thus formed will be proportional to the bases.*

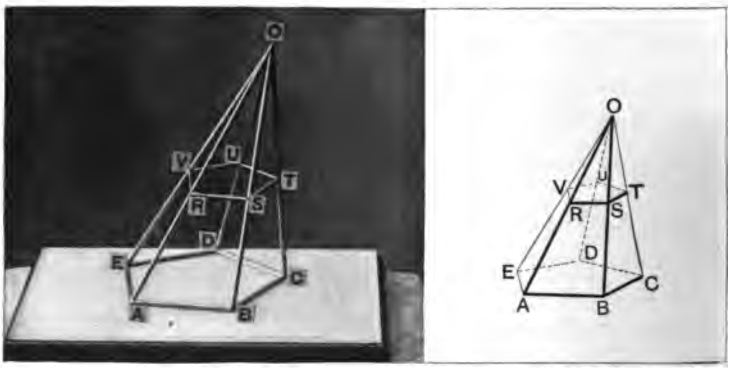


Outline proof: $$\frac{\text{area } A'B'C'D'}{\text{area } ABCD}=\frac{\overline{VO'}^2}{\overline{VO}^2}=\frac{\overline{TS'}^2}{\overline{TS}^2}=\frac{\text{area } P'K'L'}{\text{area } PKL}.$$

698. Cor. III. ***If two pyramids have equal altitudes and equivalent bases, sections parallel to their bases and equally distant from their vertices are equivalent.***

PROPOSITION XIX. THEOREM

699. ***If the lateral edges of a pyramid are divided proportionally, the points of division lie in a plane parallel to the base of the pyramid.***



Given—the pyramid $O\text{-}ABCDE$ and the points R, S, T, etc., dividing the lateral edges so that $\frac{OA}{OR}=\frac{OB}{OS}=\frac{OC}{OT}=$ etc.

To prove—that R, S, T, etc., lie in a plane parallel to the base.

Draw the straight lines RS, ST, etc.

In the triangle OAB the line RS, which divides the sides proportionally, is parallel to the base AB. § 273

Similarly ST is parallel to BC, etc.

Hence the plane of RS and ST is parallel to the base.

Similarly the plane of ST and TU is parallel to the base.

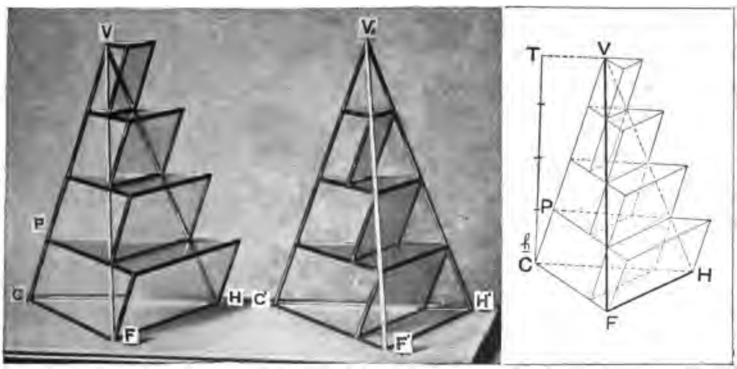
These planes coincide. § 554

In this way all the points R, S, T, etc., can be shown to lie in one plane parallel to the base. Q. E. D.

700. *Defs.*—A **triangular pyramid** is one whose base is a triangle; a **quadrangular pyramid**, one whose base is a quadrilateral.

PROPOSITION XX. THEOREM

701. *The volume of a triangular pyramid is the limit of the sum of the volumes of a series of inscribed or circumscribed prisms of equal altitude, when their number is indefinitely increased.*

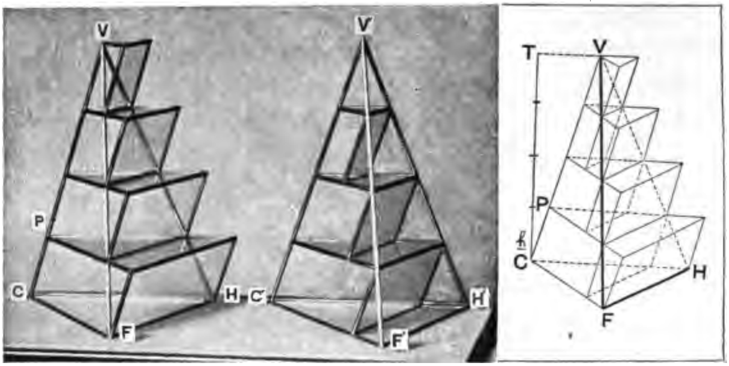


GIVEN—the triangular pyramid $V\text{-}CFH$, its altitude being CT.

TO PROVE—that its volume is the limit of the sum of the volumes of a series of inscribed or circumscribed prisms of equal altitude, when their number is indefinitely increased.

Divide the altitude CT into any number of equal parts and call one of these parts h.

Through the points of division pass planes parallel to the base, forming triangular sections. § 695 II



Upon the base *CFH* and upon the sections as *lower* bases construct prisms having their lateral edges parallel to *VC* and their altitudes equal to *h*.

This set of prisms may be said to be *circumscribed* about the pyramid.

Also with the sections as *upper* bases construct prisms having their lateral edges parallel to *VC* and their altitudes equal to *h*.

This set of prisms may be said to be *inscribed* in the pyramid.

The first circumscribed prism (beginning at the top) is equivalent to the first inscribed prism, the second circumscribed to the second inscribed, and so on until the last circumscribed remains. § 677

Hence the sum of the inscribed prisms differs from the sum of the circumscribed by the lower circumscribed prism *P-CFH.*

But the pyramid is intermediate between the total inscribed and the total circumscribed prisms. Ax. 10

Therefore the difference between the pyramid and either

of these totals is less than the difference between the totals themselves, i. e., less than the lower circumscribed prism.

But the volume of this prism is the product of its base and altitude, and since its altitude can be indefinitely diminished, while its base remains the same, its volume can be made as small as we please. § 187

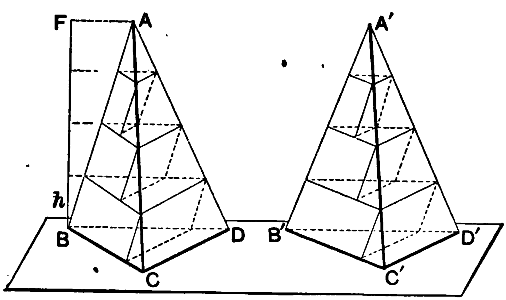
That is, the total of the inscribed prisms, or the total of the circumscribed prisms, can be made to differ from the pyramid by less than any assigned volume.

But they can never become equal to the pyramid. Ax. 10

Therefore the volume of the pyramid is their common limit. Q. E. D.

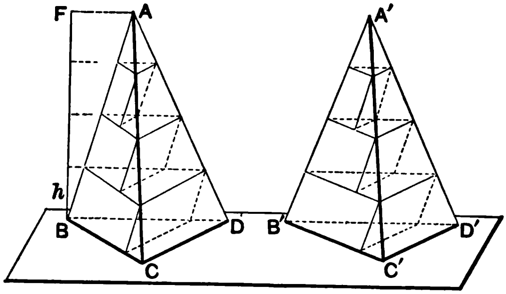
PROPOSITION XXI. THEOREM

702. ***Two triangular pyramids having equal altitudes and equivalent bases are equivalent.***



GIVEN—the triangular pyramids $A\text{-}BCD$ and $A'\text{-}B'C'D'$ having equivalent bases BCD and $B'C'D'$ in the same plane and having a common altitude BF.

TO PROVE the pyramids are equivalent.



Divide BF into any number of equal parts and denote one of these parts by h.

Through the points of division pass planes parallel to the bases and cutting the two pyramids.

The corresponding sections made by these planes in the two pyramids will be equivalent. § 698

Inscribe in each pyramid a series of prisms having the sections as upper bases and having the common altitude h.

The corresponding prisms, having equal altitudes and equivalent bases, will be equivalent. § 677

Therefore the total volume (or S) of the prisms inscribed in $A\text{-}BCD$ will equal the total volume (or S') of the prisms inscribed in $A'\text{-}B'C'D'$.

Now suppose the number of divisions of the altitude BF to be indefinitely increased.

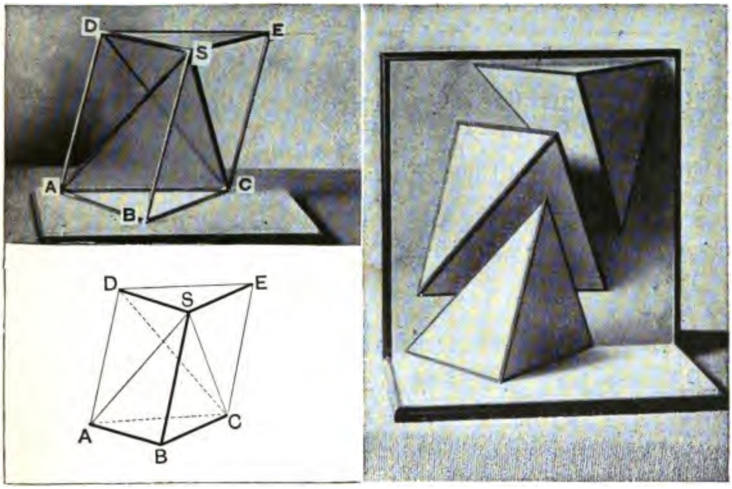
Then S will approach the volume of the pyramid $A\text{-}BCD$ as a limit, and S' will approach the volume of the pyramid $A'\text{-}B'C'D'$ as a limit. § 701

Since the variables S and S' are always equal to each other, their limits are equal. § 186

That is, the volumes of the pyramids are equal. Q. E. D.

PROPOSITION XXII. THEOREM

703. *The volume of a triangular pyramid is one-third the product of its base and altitude.*



GIVEN the triangular pyramid S-ABC.

TO PROVE—its volume is one-third its base ABC by its altitude.

Construct a triangular prism having ABC for its base and its lateral edges equal and parallel to BS.

Taking away the triangular pyramid S-ABC from the prism, we have left the quadrangular pyramid S-$DACE$.

Divide the latter by the plane SDC into two triangular pyramids S-DAC and S-DCE.

These pyramids have equal bases, the triangles DCA and DCE. § 116

They have equal altitudes, the perpendicular from the common vertex S upon the common plane of their bases.

Therefore they are equivalent. § 702

It can also be shown that the pyramids *S-ABC* and *S-DAC*, regarded as having the common vertex *C*, have equal bases and equal altitudes.

Hence these two pyramids are equivalent.

Hence all three are equivalent.

Therefore the pyramid *S-ABC* is one-third of the prism.

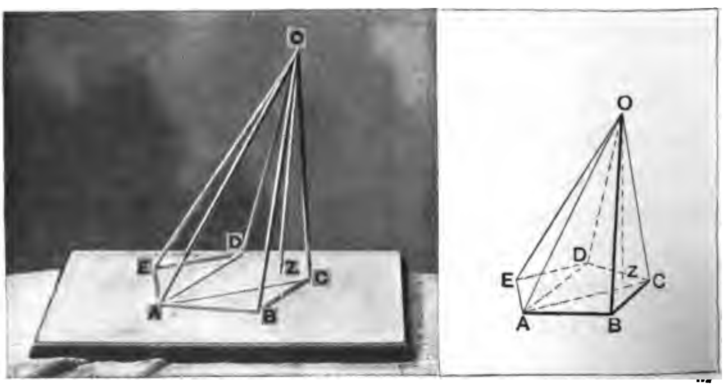
But the volume of the prism is the product of its base and altitude. § 675

And the pyramid has the same base and altitude.

Hence the volume of the pyramid is one-third the product of its base and altitude. Q. E. D.

PROPOSITION XXIII. THEOREM

704. *The volume of any pyramid is equal to one-third the product of its base and altitude.*



GIVEN the pyramid *O-ABCDE*, whose altitude is *OZ*.

TO PROVE vol. $O\text{-}ABCDE = \frac{1}{3} ABCDE \times OZ$.

Divide $ABCDE$ into triangles by diagonals drawn from A. Planes passed through OA and these diagonals will divide the pyramid into triangular pyramids, $O\text{-}ABC$, $O\text{-}ACD$, and $O\text{-}ADE$.

The volume of each triangular pyramid is one-third the product of its base and the common altitude OZ. § 703

Therefore the volume of the whole pyramid is one-third the sum of the bases of the triangular pyramids, i. e., the base of the whole pyramid multiplied by the common altitude. Q. E. D.

705. COR. I. *Pyramids having equivalent bases and equal altitudes are equivalent.*

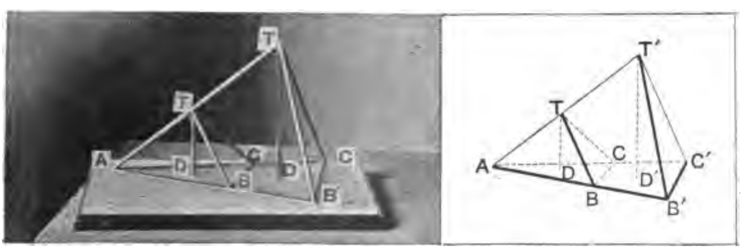
706. COR. II. *Any two pyramids are to each other as the products of their bases and altitudes.*

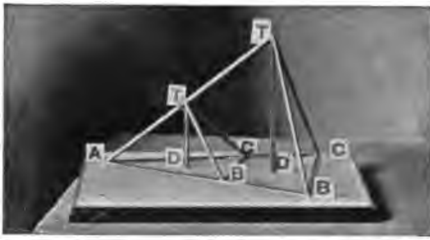
707. COR. III. *Two pyramids having equivalent bases are to each other as their altitudes.*

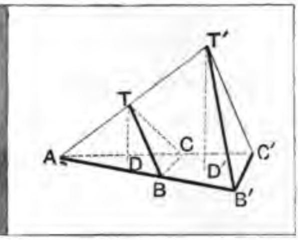
708. COR. IV. *Two pyramids having equal altitudes are to each other as their bases.*

PROPOSITION XXIV. THEOREM

709. *Two tetraedrons which have a triedral angle of one equal to a triedral angle of the other are to each other as the products of the three edges about the equal triedral angles.*







GIVEN—the tetraedrons $TABC$ and $T'A'B'C'$ having the triedral angle A in common. Let V and V' denote their respective volumes.

TO PROVE $$\frac{V}{V'}=\frac{AB\times AC\times AT}{AB'\times AC'\times AT'}.$$

From T and T' let fall the perpendiculars TD and $T'D'$ upon the plane ABC.

The three points A, D, and D' lie in one straight line. § 584

Now, considering ABC and $AB'C'$ to be the bases of the tetraedrons,

$$\frac{V}{V'}=\frac{ABC\times TD}{AB'C'\times T'D'}=\frac{ABC}{AB'C'}\times\frac{TD}{T'D'}. \qquad \S\ 706$$

But $$\frac{ABC}{AB'C'}=\frac{AB\times AC}{AB'\times AC'}. \qquad \S\ 398$$

And since TD is parallel to $T'D'$, § 563
the triangles ATD and $A'T'D'$ are similar. § 275

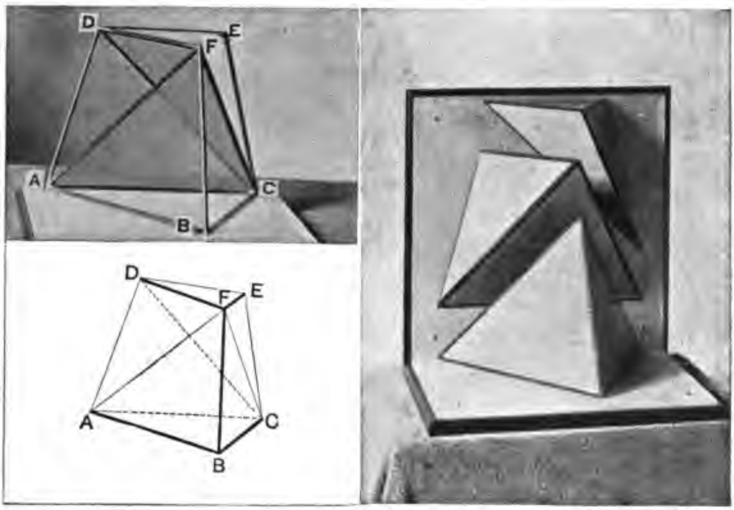
Hence $$\frac{TD}{T'D'}=\frac{AT}{A'T'}.$$

Therefore
$$\frac{V}{V'}=\frac{AB\times AC}{AB'\times AC'}\times\frac{AT}{AT'}=\frac{AB\times AC\times AT}{AB'\times AC'\times AT'}.$$ Q. E. D.

710. *Def.*—The **altitude** of a frustum of a pyramid is the perpendicular distance between the planes of its bases.

PROPOSITION XXV. THEOREM

711. *A frustum of a triangular pyramid is equivalent to the sum of three pyramids whose common altitude is the altitude of the frustum, and whose bases are the lower base, the upper base, and a mean proportional between the bases of the frustum.*



GIVEN—the frustum $ABC-DEF$ of a triangular pyramid.

TO PROVE—it is equivalent to the sum of three pyramids, etc.

Pass a plane through F, A, C, and another through F, D, C, thus dividing the frustum into three triangular pyramids, $F\text{-}ABC$, $C\text{-}DEF$, and $F\text{-}DAC$.

Call these pyramids P, Q, and R respectively, and represent ABC by B, DEF by b, and the altitude of the frustum by h.

It is evident that if B and b be taken as the bases of P and Q, they have the common altitude h. § 565

Hence $P=\frac{1}{3}h\times B$, and $Q=\frac{1}{3}h\times b$. § 703

It remains to prove that R is equivalent to a pyramid whose altitude is h and whose base is $\sqrt{B\times b}$.

The pyramids P and R, regarded as having the common vertex C and their bases in the same plane, have the same altitude.

Hence $$\frac{P}{R}=\frac{ABF}{ADF}.$$ § 708

But the triangles ABF and ADF have the same altitude, that of the trapezoid $ABFD$.

Hence $$\frac{ABF}{ADF}=\frac{AB}{DF}.$$ § 394

Hence $$\frac{P}{R}=\frac{AB}{DF}.$$ Ax. 1

Similarly $$\frac{R}{Q}=\frac{DAC}{DCE}=\frac{AC}{DE}.$$

Now the triangles ABC and DFE are similar. § 695 II

Hence $$\frac{AB}{DF}=\frac{AC}{DE}.$$ § 274

Therefore $$\frac{P}{R}=\frac{R}{Q}.$$ Ax. 1

Hence $$R^2=P\times Q.$$ § 250

Hence $R=\sqrt{P\times Q}=\sqrt{\frac{1}{3}h\times B\times\frac{1}{3}h\times b}=\frac{1}{3}h\times\sqrt{B\times b}$.

Therefore R is equivalent to a pyramid whose altitude is h and whose base is $\sqrt{B\times b}$. § 704

Q. E. D.

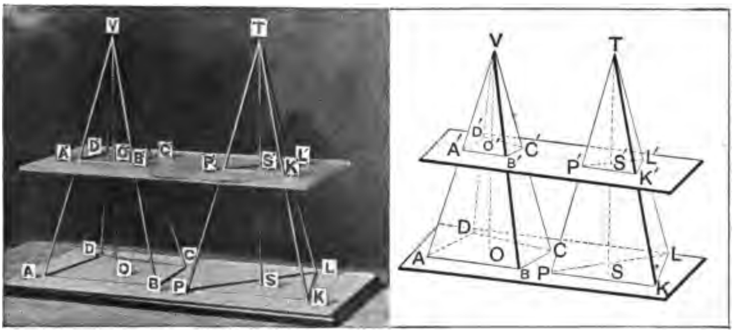
712. *Remark.*—If we denote the volume of the frustum by V, the proposition may be expressed in the form

$$V=\frac{1}{3}h(B+b+\sqrt{B\times b}).$$

Question.—Does it follow from Proposition XXV. that R *is* a pyramid whose altitude is h and base $\sqrt{B\times b}$?

PROPOSITION XXVI. THEOREM

713. *A frustum of any pyramid is equivalent to the sum of three pyramids whose common altitude is the altitude of the frustum and whose bases are the lower base, the upper base, and a mean proportional between the bases of the frustum.*



GIVEN the frustum AC' of the pyramid $V\text{-}ABCD$.

Denote its lower and upper bases by B and b respectively, its altitude by h, and its volume by V.

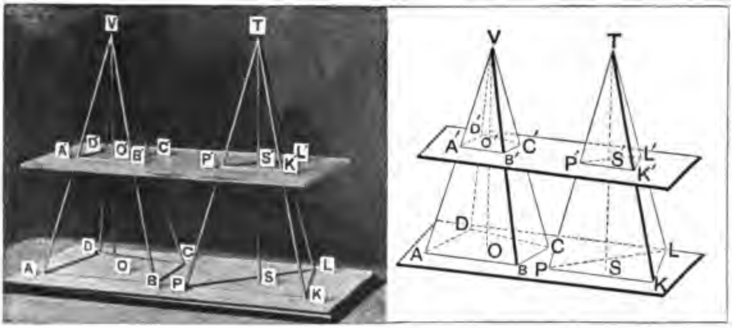
TO PROVE $V=\frac{1}{3}h(B+b+\sqrt{B\times b})$.

Let $T\text{-}PKL$ be a triangular pyramid whose base is in the same plane as $ABCD$ and equivalent to $ABCD$, whose vertex T is on the same side of this plane as V and whose altitude is equal to that of $V\text{-}ABCD$.

Prolong the plane of $A'B'C'D'$ to cut $T\text{-}PKL$ in the section $P'K'L'$.

Set B', b', h', V' for the lower base, upper base, altitude, and volume respectively of the triangular frustum PL'.

Then $V'=\frac{1}{3}h'(B'+b'+\sqrt{B'\times b'})$. (1) § 711



Now by hypothesis $B'=B$, and $h'=h$.

Moreover, $b'=b$. § 698

Again $V\text{-}ABCD$ and $V\text{-}A'B'C'D'$ are respectively equivalent to $T\text{-}PKL$ and $T'\text{-}P'K'L'$. § 705

Taking away the small pyramids, the frustums remaining are equivalent. Ax. 3

Or $V'=V$.

Substituting for V', h', B', b', their equals in (1), we get

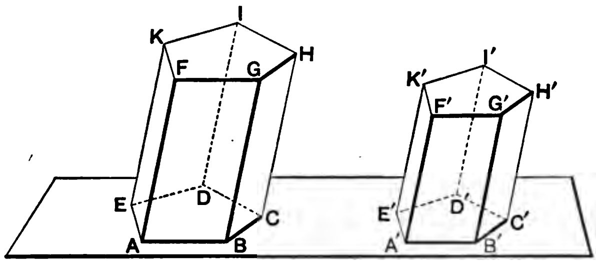
$$V=\tfrac{1}{3}h(B+b+\sqrt{B\times b}).$$ Q. E. D.

SIMILAR POLYEDRONS

714. *Def.*—Two polyedrons are **similar** if they have the same number of faces similar each to each and similarly placed, and their homologous diedral angles are equal.

PROPOSITION XXVII. THEOREM

715. *The ratio of any two homologous edges of two similar polyedrons is equal to the ratio of any other two homologous edges.*



GIVEN—the similar polyedrons AH and $A'H'$ in which any two edges AB and CH of one are respectively homologous to $A'B'$ and $C'H'$ of the other.

TO PROVE $$\frac{AB}{A'B'}=\frac{CH}{C'H'}.$$

Since the faces $ABGF$ and $A'B'G'F'$ are similar,

$$\frac{AB}{A'B'}=\frac{BG}{B'G'}. \quad \S\ 274$$

Since the faces $BCHG$ and $B'C'H'G'$ are similar,

$$\frac{CH}{C'H'}=\frac{BG}{B'G'}.$$

Therefore $$\frac{AB}{A'B'}=\frac{CH}{C'H'}. \quad \text{Ax. 1}$$

Q. E. D.

716. *Def.*—The ratio of any two homologous edges of two similar polyedrons is called the **ratio of similitude** of the polyedrons.

717. COR. I. *The ratio of any two homologous faces of two similar polyedrons is equal to the square of their ratio of similitude.*

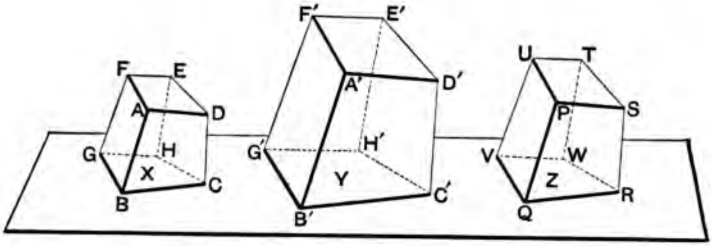
Hint.—Apply § 401.

718. COR. II. *The ratio of the total surfaces of two similar polyedrons is equal to the square of their ratio of similitude.*

Hint.—Apply § 265.

PROPOSITION XXVIII. THEOREM

719. *Two polyedrons similar to a third are similar to each other.*



GIVEN—the polyedrons AH, or X, and $A'H'$, or Y, both similar to PW, or Z.

TO PROVE that X is similar to Y.

The faces AC and $A'C'$, being both similar to PR, are similar to each other. § 294

In the same way all the faces of X may be shown similar to corresponding faces of Y.

The polyedrons X and Y also have the similar faces similarly arranged, since the arrangement in each is the same as the arrangement in Z.

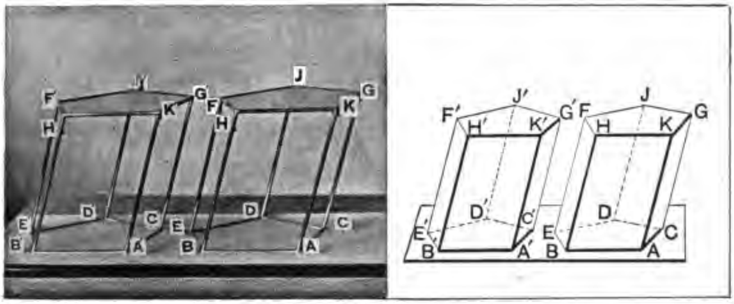
Lastly, any two homologous diedral angles of X and Y, being each equal to the same diedral angle of Z, are equal to each other. Ax. 1

Therefore the polyedrons X and Y are similar. § 714

Q. E. D.

PROPOSITION XXIX. THEOREM

720. *Two similar polyedrons are equal, if their ratio of similitude is unity.*



GIVEN—the similar polyedrons AF and $A'F'$ having

$$\frac{AB}{A'B'}=\frac{CG}{C'G'}=\text{etc.}=1.$$

TO PROVE these polyedrons are equal.

The homologous faces are equal, being similar and having unity as a ratio of similitude. § 296

Superpose the faces $ABHK$ and $A'B'H'K'$.

Since the diedral angles AK and $A'K'$ are equal, the planes of the faces $CAKG$ and $C'A'K'G'$ will coincide, and since the side AK of one already coincides with the side $A'K'$ of the other, these faces, being equal, will coincide throughout.

In this way all the faces can be shown to coincide.

Therefore the polyedrons are equal. Q. E. D.

PROPOSITION XXX. THEOREM

721. *If two diedral angles have their faces respectively parallel and extending in the same direction, they are equal.*

The proof is left to the student.

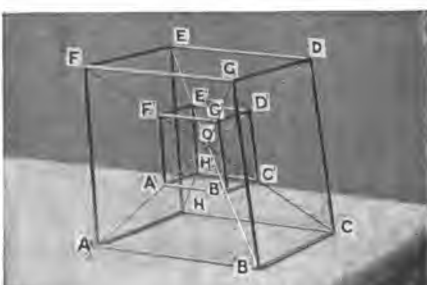
722. *Defs.*—If the vertices A, B, C, D, etc., of a polyedron are joined by straight lines to any point O, and the lines OA, OB, OC, OD, etc., are divided in the same ratio at the points A', B', C', D', etc., the polyedron $A'B'C'D'$, etc., is said to be **radially situated** with regard to the polyedron $ABCD$, etc.

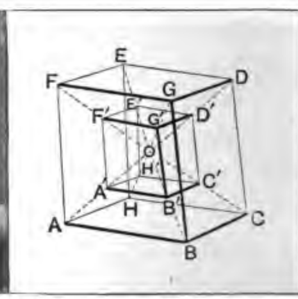
The ratio of the rays OA' and OA is called the **determining ratio**, or **ray ratio**, of the two polyedrons.

The point O is called the **ray centre.**

PROPOSITION XXXI. THEOREM

723. *Two radially situated polyedrons are similar, and their ratio of similitude is equal to the ray ratio.*





GIVEN—the radially situated polyedrons AD and $A'D'$, O being the ray centre.

TO PROVE—that they are similar, and that the ray ratio is their ratio of similitude.

The two polyedrons are made up of pyramids having O for common vertex.

In the pyramid $O\text{-}ABCH$ the plane $A'C'$ is parallel to the base. § 699

Hence the polygons $A'C'$ and AC are similar. § 695 II

In the same way all the faces of one polyedron can be shown to be similar to the corresponding faces of the other.

And the diedral angles are equal, since their faces are respectively parallel and extending in the same direction from their edges. § 721

Therefore the polyedrons are similar. § 714

Again, the triangles $OA'B'$ and OAB are similar. § 285

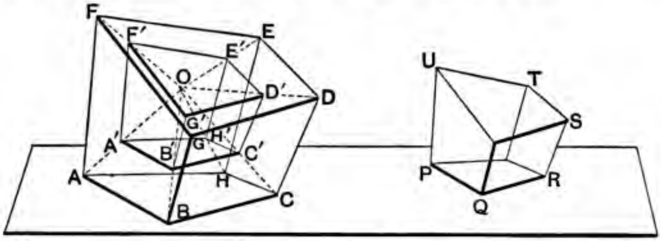
Hence the ratio of similitude of the polyedrons, $\frac{A'B'}{AB}$, is equal to the ray ratio, $\frac{OA'}{OA}$. Q. E. D.

724. *Remark.*—The student should draw a figure with the ray centre outside of the polyedrons and show that the proof is the same in this case. He should also draw a figure in which the ray centre is a common vertex of the two polyedrons. The proof is slightly different with such a figure.

725. *Def.*—The ray centre is also called the **centre of similitude.**

PROPOSITION XXXII. THEOREM

726. *Any two similar polyedrons can be radially placed, the ray ratio being equal to the ratio of similitude.*



GIVEN the similar polyedrons FC and UR.

TO PROVE—that they can be radially placed, the ray ratio being the ratio of similitude.

With any point O as ray centre form a polyedron $F'C'$ radially situated with regard to FC, having the ray ratio $\frac{OA'}{OA}$ equal to the ratio of similitude $\frac{PQ}{AB}$ of UR and FC.

Then $F'C'$ and FC will be similar, the ratio of similitude $\frac{A'B'}{AB}$ being equal to the ray ratio $\frac{OA'}{OA}$. § 723

But UR and FC are given similar, and their ratio of similitude is $\frac{PQ}{AB}$.

Therefore $F'C'$ and UR are similar. § 719

Now since $\frac{A'B'}{AB}=\frac{OA'}{OA}$, and $\frac{OA'}{OA}=\frac{PQ}{AB}$,

then $$\frac{A'B'}{AB}=\frac{PQ}{AB}.$$

By alternation $$\frac{A'B'}{PQ}=\frac{AB}{AB}=1.$$

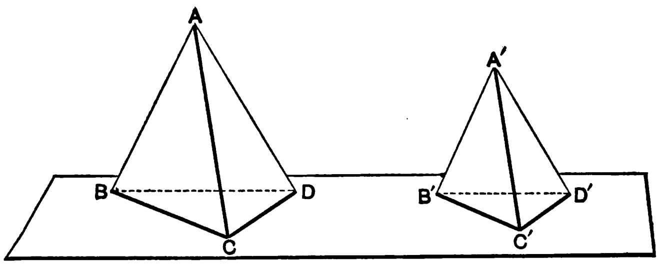
That is, the ratio of similitude of $F'C'$ and UR is unity.

Therefore UR can be made to coincide with $F'C'$. § 720

In other words, FC and UR can be radially placed, the ray ratio being the ratio of similitude. Q. E. D.

PROPOSITION XXXIII. THEOREM

727. *Two similar tetraedrons are to each other as the cubes of any two homologous edges.*



GIVEN the similar tetraedrons $ABCD$ and $A'B'C'D'$.

TO PROVE $\dfrac{\text{vol. } ABCD}{\text{vol. } A'B'C'D'} = \dfrac{\overline{AB}^3}{\overline{A'B'}^3}$.

The face angles of the triedral angles A and A' are equal each to each and similarly arranged. § 274

Hence these triedral angles are equal. § 597

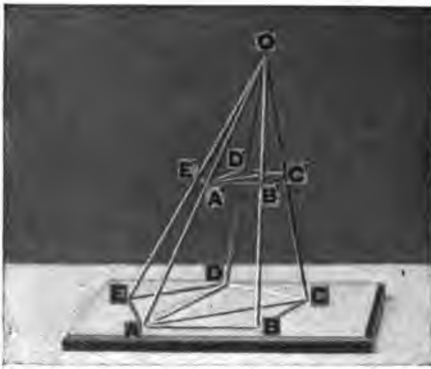
Therefore $\dfrac{\text{vol. } ABCD}{\text{vol. } A'B'C'D'} = \dfrac{AB \times AC \times AD}{A'B' \times A'C' \times A'D'}$ § 709

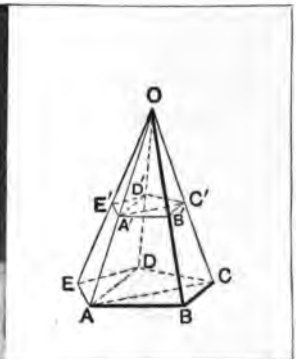
$$= \frac{AB}{A'B'} \times \frac{AC}{A'C'} \times \frac{AD}{A'D'}$$

$$= \frac{AB}{A'B'} \times \frac{AB}{A'B'} \times \frac{AB}{A'B'}. \quad \text{§ 715}$$

That is, $\dfrac{\text{vol. } ABCD}{\text{vol. } A'B'C'D'} = \dfrac{\overline{AB}^3}{\overline{A'B'}^3}$. Q. E. D.

728. COR. *If a pyramid is cut by a plane parallel to its base, the pyramid cut off is similar to the first, and the two pyramids are to each other as the cubes of any two homologous edges.*





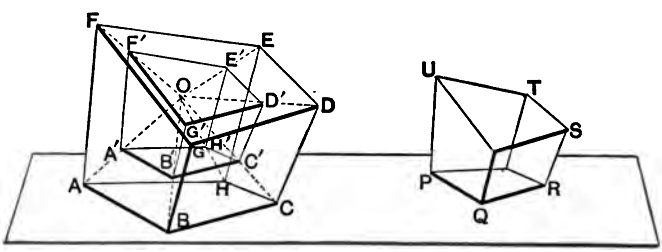
Hint.—Prove first that the lateral edges are divided proportionally. The pyramids are therefore similar (§ 723). Divide the pyramids into similar triangular pyramids as shown in the figure.

Then $\dfrac{\overline{OA}^3}{\overline{OA'}^3} = \dfrac{\text{vol. } O\text{-}ABC}{\text{vol. } O\text{-}A'B'C'} = \dfrac{\text{vol. } O\text{-}ACD}{\text{vol. } O\text{-}A'C'D'} = \dfrac{\text{vol. } O\text{-}ADE}{\text{vol. } O\text{-}A'D'E'}.$

Now apply § 265.

PROPOSITION XXXIV. THEOREM

729. *The ratio of the volumes of any two similar polyedrons is equal to the cube of their ratio of similitude.*



GIVEN the two similar polyedrons FC and UR.

TO PROVE $\dfrac{\text{vol. } FC}{\text{vol. } UR} = (\text{ratio of similitude})^3$.

Suppose that UR is smaller than FC.

Place UR within FC in the position $F'C'$, radially situated with regard to FC, the ray centre being O. § 726

Then each polyedron can be divided into pyramids having O for common vertex and the faces of the polyedron for bases.

The planes $A'B'C'H'$, $B'C'D'G'$, etc., are respectively parallel to the planes $ABCH$, $BCDG$, etc. § 699

Hence

$$\frac{\text{vol. } O\text{-}ABCH}{\text{vol. } O\text{-}A'B'C'H'} = \frac{\overline{OA}^3}{\overline{OA'}^3} = \frac{\overline{OB}^3}{\overline{OB'}^3} = \frac{\text{vol. } O\text{-}BCDG}{\text{vol. } O\text{-}B'C'D'G'} = \text{etc.}$$

§ 728

Therefore

$$\frac{\text{vol. } O\text{-}ABCH + \text{vol. } O\text{-}BCDG + \text{etc.}}{\text{vol. } O\text{-}A'B'C'H' + \text{vol. } O\text{-}B'C'D'G' + \text{etc.}} = \frac{\overline{OA}^3}{\overline{OA'}^3}. \quad \text{§ 265}$$

That is,

$$\frac{\text{vol. } FC}{\text{vol. } UR} = \frac{\overline{OA}^3}{\overline{OA'}^3}.$$

$$= (\text{ray ratio})^3$$

$$= (\text{ratio of similitude})^3. \quad \text{§ 726}$$

Q. E. D.

REGULAR POLYEDRONS

730. *Def.*—A **regular polyedron** is one whose faces are equal regular polygons, and whose diedral angles are all equal.

PROPOSITION XXXV. THEOREM

731. *Not more than five regular convex polyedrons are possible.*

Proof.—The faces of a regular polyedron must be regular polygons; at least three faces are necessary to form a polyedral angle; and the sum of the face angles of a convex polyedral angle must be less than 360°. Hence,

Firstly, since each angle of an *equilateral triangle* is equal to 60° (§ 64), either three, four, or five equilateral triangles can be combined to form a convex polyedral angle.

Not more than five equilateral triangles can be so combined, for six would make the sum of the face angles $6 \times 60° = 360°$, which is impossible.

Therefore not more than three regular convex polyedrons can be formed with triangular faces.

Secondly, since each angle of a *square* is equal to 90° (§ 114), three squares can be combined to form a convex polyedral angle.

Not more than three squares can be so combined, since $4 \times 90° = 360°$.

Therefore not more than one regular convex polyedron can be formed with square faces.

Thirdly, since each angle of a *regular pentagon* is equal to 108° (§ 463), three regular pentagons can be combined to form a convex polyedral angle.

Not more than three regular pentagons can be so combined, since $4 \times 108° = 432°$.

Therefore not more than one regular convex polyedron can be formed with pentagonal faces.

Fourthly, since each angle of a *regular hexagon* is equal to 120° (§ 463), no convex polyedral angle can be formed with regular hexagons, for $3 \times 120° = 360°$.

Similarly it can be shown that no convex polyedral angle can be formed with regular polygons of *more* than six sides.

Therefore not more than five regular convex polyedrons can be formed. Q. E. D.

732. We will now show by actual construction that exactly five regular convex polyedrons can be formed, viz.:

(1.) The regular tetraedron, whose four faces are equilateral triangles.

(2.) The regular hexaedron, or cube, whose six faces are squares.

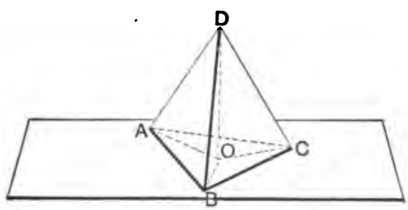
(3.) The regular octaedron, whose eight faces are equilateral triangles.

(4.) The regular dodecaedron, whose twelve faces are regular pentagons.

(5.) The regular icosaedron, whose twenty faces are equilateral triangles.

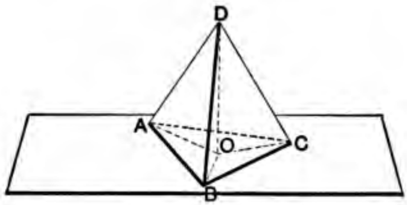
ICOSAEDRON DODECAEDRON OCTAEDRON HEXAEDRON TETRAEDRON

733. CONSTRUCTION. *To construct a regular tetraedron.*



Construct the equilateral triangle ABC.

At the centre O of the circumscribing circle erect the perpendicular OD to the plane ABC.



Take D so that $AD = AB$, and draw AD, BD, CD, and OA, OB, and OC.

TO PROVE that $ABCD$ is a regular tetraedron.

Since O is the centre of ABC,

$$OA = OB = OC.$$

Therefore $$AD = BD = CD.$$ § 539 I

But AD was constructed equal to AB, and AB was given equal to BC and AC.

The four faces are therefore equal equilateral triangles.

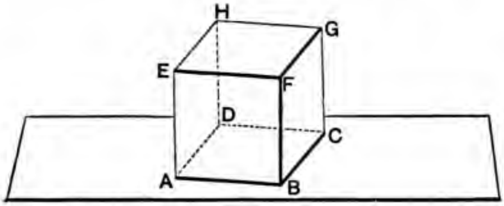
Also, since all the face angles of the four triedral angles are equal, all the triedral angles are equal. § 597

By superposing the triedral angles in pairs it may be seen that all the diedral angles are equal.

Therefore $ABCD$ is a regular tetraedron. § 730

Q. E. F.

734. CONSTRUCTION. *To construct a regular hexaedron.*



Draw the three equal straight lines AB, AD, and AE perpendicular to each other.

Upon them construct a rectangular parallelopiped AG. § 673

The faces will all be squares. § 114

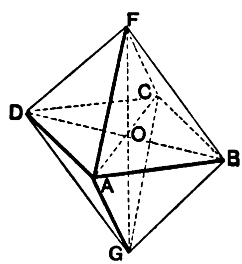
They will all be equal. § 377

That is, all the faces of the polyedron AG are equal regular quadrilaterals.

Its diedral angles are all equal, since their plane angles are right angles. § 572

Therefore AG is a regular hexaedron. Q. E. F.

735. CONSTRUCTION. *To construct a regular octaedron.*



Construct the square $ABCD$.

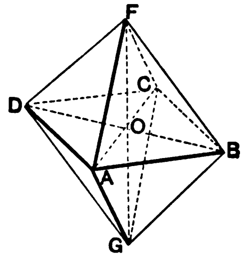
Through its centre O draw the straight line FG perpendicular to its plane.

In FG take two points F and G so that $OF=OG=OB$.

Join F and G to the points A, B, C, D.

TO PROVE that $FABCDG$ is a regular octaedron.

The angles AOB and AOF are right angles and $AO=OB=OF$.



Therefore the triangles AOB and AOF are equal.

Hence $AF = AB$.

Also $AF = BF = CF = DF = AG = BG = CG = DG$.

The eight faces are therefore equal equilateral triangles.

Again, by construction FG and DB are equal and bisect each other at right angles.

Therefore $DFBG$ is a square.

It is equal to $ABCD$ and AO is perpendicular to its plane.

Hence the pyramids A-$DFBG$ and F-$ABCD$ are superposable and from symmetry each of the diedral angles AD, AF, AB, AG can be made to coincide with each of the diedral angles FA, FB, FC, FD.

Similarly any two diedral angles can be shown to be equal.

Therefore $FABCDG$ is a regular octaedron. Q. E. F.

736. CONSTRUCTION. *To construct a regular dodecaedron.*

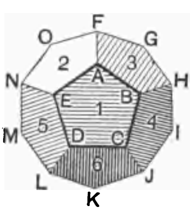


FIG. 1

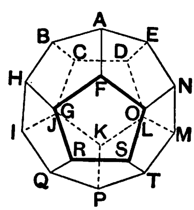


FIG. 2

Combine three equal regular pentagons *ABCDE*, *AENOF*, *AFGHB* so as to form a triedral angle at *A* (Fig. 1).

Pass a plane through *H*, *B*, and *C*.

There will then be formed at *B* a triedral angle equal to that at *A*.

For, the diedral angle *AB* is common, and the face angles *CBA* and *HBA* of *B* are equal to the face angles *FAB* and *EAB* of *A*.

Hence the angle *HBC* is equal to an angle of a regular pentagon.

We can add, therefore, to the three pentagons already united a fourth regular pentagon, *HBCJI* having *HBC* for an angle.

Similarly we can add the fifth regular pentagon *NEDLM*.

Now the triedral angles at *D* and *C* can be shown to be equal to that at *A* by the process used above.

Hence the angles *CDL* and *DCJ* are each equal to an angle of a regular pentagon.

And *LD*, *DC*, and *CJ* are in the same plane.

For the plane of *LD* and *DC* forms a diedral angle with face *1* at *DC* equal to that at *AB*, and the plane of *DC* and *CJ* forms a diedral angle with face *1* at *DC* equal to that at *AB*, and therefore these planes coincide.

We can therefore add to the five pentagons already joined a sixth regular pentagon *LDCJK* having *LD*, *DC*, and *CJ* as sides.

Now, as we added the fourth pentagon, so we can add the seventh *FOSRG* (Fig. 2).

As we added the sixth, so we can add successively the eighth, ninth, and tenth *OSTMN*, *MTPKL*, *KPQIJ*.

Now, as we showed that the lines *LD*, *DC*, and *CJ* were

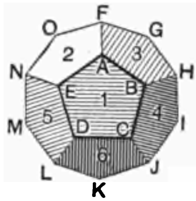


FIG. 1

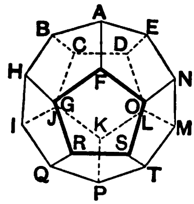


FIG. 2

in one plane, so we can show that the plane of *GH* and *HI* contains the lines *GR* and *IQ*.

The angles these lines make with each other can be shown as above equal to an angle of a regular pentagon.

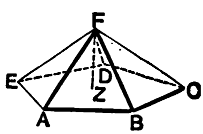
We can therefore add the eleventh regular pentagon *RGHIQ*.

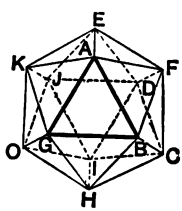
The twelfth pentagon can by the same methods be shown to be regular and equal to the others.

The diedral angles are all easily seen to be equal.

Therefore the polyedron thus formed is a regular dodecaedron. Q. E. F.

737. CONSTRUCTION. *To construct a regular icosaedron.*





Construct the regular pentagon *ABCDE*.

At its centre *Z* erect the perpendicular *ZF* to its plane, making $AF=AB$. Draw *AF*, *BF*, *CF*, *DF*, *EF*.

Then *F-ABCDE* is a regular pyramid and its five lateral faces are equal equilateral triangles.

Form nine other pyramids equal to $F\text{-}ABCDE$.

Now one of these can be made to coincide with $F\text{-}ABCDE$ in five different ways. For it makes no difference which side of its base coincides with AB.

Hence all of the diedral angles FA, FB, etc., are equal to any one of the diedral angles of the second pyramid, and are therefore all equal to each other.

Now place one of the seven pyramids, say $A'\text{-}B'F'E'KG$, so that the diedral angle $A'F'$ shall coincide with its equal AF and the faces $A'F'B'$ and $A'F'E'$ with their equals AFB and AFE; thus adding the new faces EAK, KAG, and GAB.

Place a second pyramid, $B'\text{-}C'F'A'G'H$, so that the diedral angles $B'F'$ and $B'A'$ shall coincide with their equals BF and BA, and the faces $B'C'F'$, $B'F'A'$, and $B'A'G'$ with their equals BCF, BFA, and BAG; thus adding the new faces GBH and HBC.

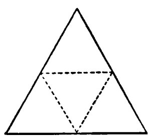
Similarly place two others, $C'\text{-}D'\ F'\ B'\ H'\ I$ and $D'\text{-}E'\ F'\ C'\ I'\ J$, with their vertices at C and D; thus adding the new faces HCI, ICD and IDJ, JDE.

Place a fifth, $E'\text{-}K'A'F'D'J'$, so that the diedral angles $E'A'$, $E'F'$, and $E'D'$ shall coincide with their equals EA, EF, and ED, and the faces $E'A'K'$, $E'F'A'$, $E'D'F'$, and $E'J'D'$ with their equals EAK, EFA, EDF, and EJD; thus adding the new face JEK.

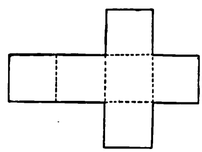
The four other pyramids can be similarly placed with their vertices at G, H, I, and J; thus adding the new faces OGK and OGH, OHI, OIJ, and OJK.

The polyedron thus completed, having twenty equal equilateral triangles for faces and having its diedral angles all equal, is a regular icosaedron. Q. E. F.

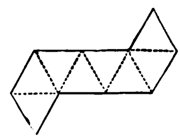
738. *Remark.*—The five regular polyedrons may be made from cardboard as follows: Draw on cardboard the figures given below, and on the inner lines cut the cardboard half through with a penknife. Cut the figures out entire and fold the cardboard as shown for the icosaedron in the accompanying plate.



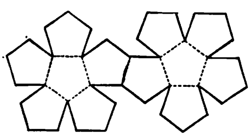
TETRAEDRON



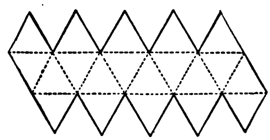
HEXAEDRON



OCTAEDRON



DODECAEDRON



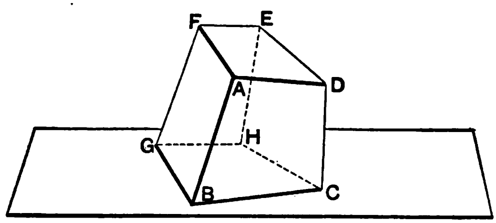
ICOSAEDRON

ICOSAEDRON

GENERAL THEOREMS ON POLYEDRONS

PROPOSITION XXXVI. THEOREM

739. *The number of the edges of any polyedron increased by two is equal to the number of its vertices increased by the number of its faces.**



GIVEN any polyedron AH.

Denote the number of its edges by E; the number of its vertices by V; and the number of its faces by F.

TO PROVE $E + 2 = V + F$.

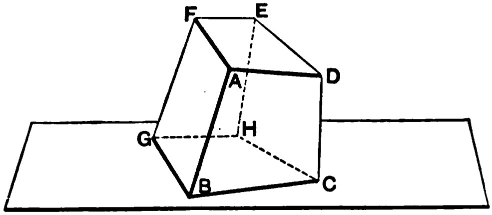
Let us put together the surface of the polyedron face by face and compare the number of edges with the number of vertices at each step.

If we take one face, as $ABCD$, the number of edges is obviously equal to the number of vertices.

That is, for one face, $E = V$.

Now let us add a second face, say a quadrilateral $ABGF$, to the first by placing the edges AB together. The new surface, consisting of $ABCD$ and $ABGF$, will have three new edges, AF, FG, and GB, and two new vertices, F and G.

* This theorem was discovered by Euler (1707–1783).



The whole number of edges will be then one greater than the whole number of vertices.

However many sides the second face may have, it is easily seen that the number of new edges added will be one more than the number of new vertices.

Therefore for two faces, $E=V+1$.

Next add a third face $ADEF$ by placing an edge of it in coincidence with an edge of each of the first two faces.

We thus add two new edges, DE and FE, and one new vertex, E.

However many sides the third face may have, the increase in the number of edges is one more than the increase in the number of vertices.

Hence for three faces, $E=V+2$.

We can in this way prove the following table:

For 1 face	$E=V$.
For 2 faces	$E=V+1$.
For 3 faces	$E=V+2$.
For m faces	$E=V+(m-1)$.
For $F-1$ faces	$E=V+F-2$.

When the number of faces is $F-1$, the surface is not closed.

To close it we add the last face.

In so doing we place each edge and each vertex of the

last face in coincidence with an edge and vertex of the open surface.

Adding the last face then increases neither the number of edges nor the number of vertices.

That is, for F faces, $E=V+F-2$.

or $$E+2=V+F.$$ Q. E. D.

PROPOSITION XXXVII. THEOREM

740. *The sum of the angles of all the faces of any convex polyedron is equal to four right angles taken as many times as the polyedron has vertices less two.*

Let S denote the sum of the angles of all the faces, and V the number of vertices of any convex polyedron. Also let R denote a right angle.

To prove $$S=4R(V-2).$$

Any one face is a convex polygon.

Let the number of its sides be n.

Produce the sides in succession as in § 69.

The sum of the exterior angles thus formed is $4R$.

The sum of the interior and exterior angles is $2R\times n$. § 22

Do the same for all the faces of the polyedron considered as independent polygons of n, n', n'', etc., sides.

Then the sum of the exterior angles of the F faces is $4R\times F$.

The sum S of their interior angles *plus* the sum of their exterior angles is $2R(n+n'+n''+\text{etc.})$.

That is, $S+4R\times F=2R(n+n'+n''+\text{etc.})$.

Now, if E denotes the number of edges of the polyedron,

$$n+n'+n''+\text{etc.}=2E,$$

since each edge is a side of two polygons.

Hence $$S+4R\times F=2R\times 2E,$$
or $$S=4R(E-F).$$

But by Euler's Theorem
$$E-F=V-2.$$

Therefore $$S=4R(V-2).$$ Q. E. D

PROBLEMS OF DEMONSTRATION

741. *Exercise.*—The four diagonals of a parallelopiped bisect each other.

742. *Exercise.*—Any straight line drawn through the intersection of the diagonals of a parallelopiped and terminated by two opposite faces is bisected in that point.

743. *Exercise.*—The sum of the squares of the four diagonals of a parallelopiped is equal to the sum of the squares of its twelve edges.

744. *Exercise.*—In a rectangular parallelopiped, the four diagonals are equal to each other; and the square of a diagonal is equal to the sum of the squares of the three edges which meet at a common vertex.

745. *Exercise.*—In a quadrangular prism the two diagonals which connect either pair of opposite edges bisect each other.

746. *Exercise.*—In any quadrangular prism the sum of the squares of the four diagonals plus eight times the square of the straight line joining the common middle points of the pairs of diagonals which bisect each other is equal to the sum of the squares of the twelve edges.

747. *Exercise.*—If a plane parallel to two opposite edges of a tetraedron cut the tetraedron, the section is a parallelogram.

748. *Exercise.*—If the angles at the vertex of a triangu-

lar pyramid are right angles, and the lateral edges are equal, prove that the sum of the perpendiculars on the lateral faces from any point in the base is constant.

749. *Exercise.*—The straight lines joining each vertex of a tetraedron with the intersection of the medians of the opposite face, meet in a point which divides each line into segments whose ratio is 3:1.

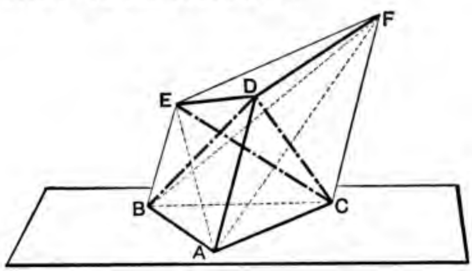
This point is called in Physics *the centre of gravity* of the tetraedron.

750. *Exercise.*—The straight lines joining the middle points of the opposite edges of a tetraedron meet in a point and are each bisected by the point.

751. *Exercise.*—A plane bisecting two opposite edges of a tetraedron divides the tetraedron into two equivalent polyedrons.

752. *Exercise.*—The pyramid whose base is one of the faces of a cube, and whose vertex is at the centre of the cube, is one-sixth part of the cube.

753. *Exercise.*—A truncated triangular prism is equivalent to the sum of three pyramids whose common base is either base of the truncated prism, and whose vertices are the three vertices of the other base.



Hint.—Divide the truncated triangular prism into three triangular pyramids by the planes *DBC* and *DEC*. Show that the pyramids *D-BEC* and *E-ABC* are equivalent. Also the pyramids *D-CEF* and *F-ABC*.

754. *Exercise.*—The volume of a right truncated triangular prism (Fig. 1) is equal to the product of one-third the sum of its lateral edges by the area of the base to which those edges are perpendicular.

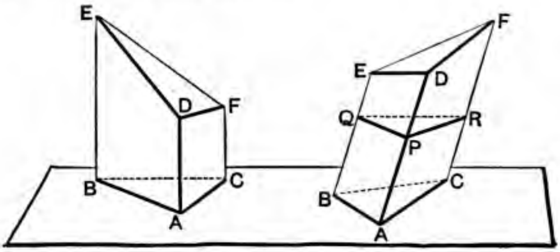


FIG. 1 FIG. 2

755. *Exercise.*—The volume of any truncated triangular prism (Fig. 2) is equal to the product of one-third the sum of its lateral edges by the area of a right section.

756. *Exercise.*—The volume of a truncated prism, one of whose bases is a parallelogram, is equal to the product of a right section by one-fourth the sum of the lateral edges.

757. *Exercise.*—The volume of a truncated triangular prism is equal to the product of the lower base by the perpendicular on the lower base from the intersection of the medians of the upper base.

758. *Exercise.*—The perpendicular from a vertex of a regular tetraedron on the opposite face is three times the perpendicular from its own foot on any of the other faces.

PROBLEMS OF CONSTRUCTION

759. *Exercise.* — Pass a plane through a straight line given in position which shall divide a given parallelopiped into two equivalent polyedrons.

760. *Exercise.*—Cut a cube by a plane so that the section shall be a regular hexagon.

761. *Exercise.*—Pass a plane through a given straight line which shall divide a given triangular prism into two equivalent truncated prisms.

762. *Exercise.*—Construct a parallelopiped of which three edges lie upon three given straight lines in space.

763. *Exercise.*—Pass a plane through a given point which shall divide a given tetraedron into two equivalent parts.

PROBLEMS FOR COMPUTATION

764. (1.) A rectangular block of marble is 1 m. 9 dcm. long, 9 dcm. 6 cm. broad, and 8 dcm. 9 cm. thick. What is its weight, if a cubic meter weighs 2675 kg.?

(2.) A barn with a gable roof is 60 ft. long, 30 ft. broad; the height from the floor to the eaves is 25 ft., to the gable $32\frac{1}{2}$ ft. Find its contents.

(3.) The area of the base of a right prism is 12 sq. in., its total area is 295 sq. in.; the base is a regular hexagon. What is the volume?

(4.) The great pyramid is estimated to have cost ten dollars a cubic yard, and three dollars besides for each square yard of surface; in this estimate the lateral faces are considered to be planes. The altitude of the pyramid is 488 ft., its base is 764 ft. square. What was its cost?

(5.) Express the volume of a cube in terms of the length of a diagonal.

(6.) What is the ratio of an edge of a cube to that of a regular tetraedron of the same volume?

(7.) The area of the lower base of a frustum of a pyramid is 100 sq. cm., of the upper base 30 sq. cm., and the altitude of the frustum is 5 dcm. What would be the altitude of the complete pyramid?

(8.) What is the volume of a frustum of a regular triangular pyramid, if its slant height is 3.5 ft., a side of the lower base 4 ft., of the upper base 1.5 ft.?

(9.) The total surface of a regular tetraedron is 400 sq. ft. What is its volume?

(10.) The area of a face of a regular octaedron is 1 sq. ft. What is its volume?

(11.) What is the ratio of the lateral area of a regular tetraedron to the lateral area of a prism constructed upon the same base and having one of its lateral edges coincident with an edge of the tetraedron?

(12.) Find the volume of a truncated triangular prism, if the sides of a right section are respectively 2.416, 3.213, 1.963 in., and its lateral edges are 7.645, 6.633, 2.742 in.

GEOMETRY OF SPACE

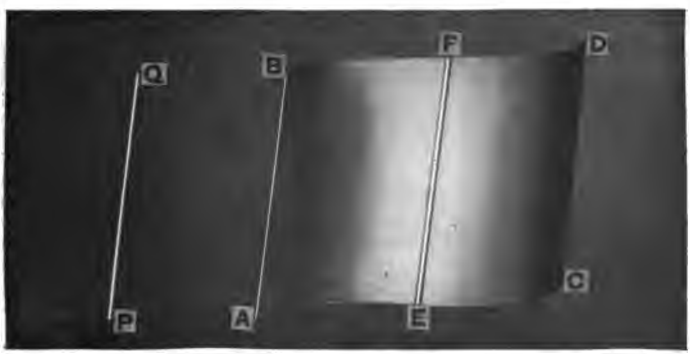
BOOK VIII

THE CYLINDER

765. *Def.*—A **curved line,** or **curve,** is a line no part of which is straight.

The curve may or may not lie entirely in one plane. An example of the first kind is the circumference of a circle; an example of the second kind is a curve like a corkscrew.

766. *Def.*—A **cylindrical surface** is a surface generated by a moving straight line which continually intersects a given fixed curve and is constantly parallel to a given fixed straight line.



Thus, if the straight line AB moves so as continually to intersect the curve AC and remains parallel to the line PQ, the surface generated, $ABDC$, is a cylindrical surface.

767. *Defs.*—The moving line is called the **generatrix;** the fixed curve is called the **directrix.**

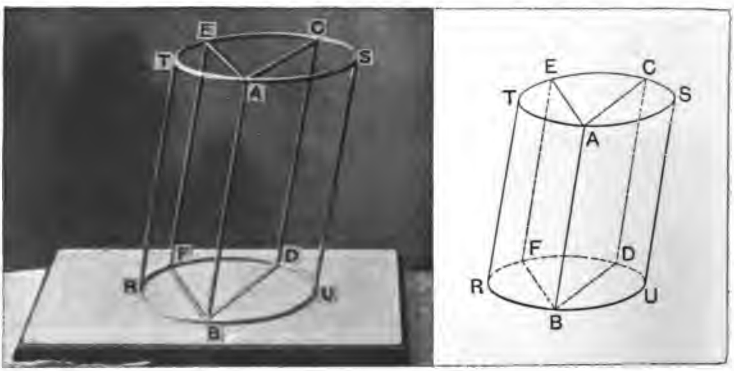
Any one position of the generatrix, as EF, is called an **element** of the surface.

768. *Remark.*—The generatrix is usually supposed to be indefinite in extent, so that the surface generated is also of indefinite extent.

The directrix may be any curve whatever. But for the student who has not studied the appendix the proofs are rigorous only when the directrix is considered to be the circumference of a circle.

PROPOSITION I. THEOREM

769. *The sections of a closed cylindrical surface made by two parallel planes cutting the elements are equal.*



GIVEN—the closed cylindrical surface RS cut by two parallel planes, not parallel to the elements, in the sections TS and RU.

TO PROVE that TS and RU are equal.

Let A, C, and E be any three points in the perimeter of the upper section, and AB, CD, and EF the corresponding

elements; B, D, and F being the points where these elements meet the perimeter of the lower section.

Through AB and CD pass a plane. Pass another through AB and EF.

Then AC is parallel to BD and AE to BF. § 544

Hence $AC=BD$ and $AE=BF$. § 118

The angles CAE and DBF are also equal. § 557

If, therefore, the planes of the two sections be superposed so that BD shall coincide with AC, F will fall on E.

Now, if we suppose AC to be fixed and the point E to describe the perimeter of the upper section, then F will describe the perimeter of the lower section.

But in the superposed position of the sections F would always coincide with E.

Hence the perimeters of the two sections would coincide throughout. Therefore the sections are equal. Q. E. D.

770. *Defs.*—A **cylinder** is a solid bounded by a closed cylindrical surface and two parallel planes.

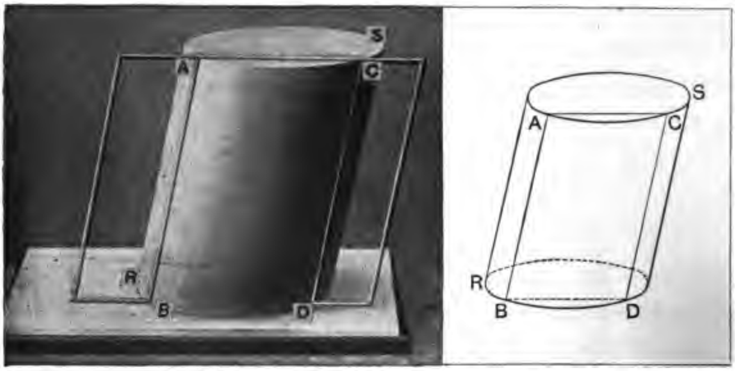
The cylindrical surface is called the **lateral surface,** and the equal sections formed by the parallel planes the **bases** of the cylinder.

CYLINDERS

The term **element of a cylinder** is used to signify an element of its lateral surface.

PROPOSITION II. THEOREM

771. *Every section of a cylinder made by a plane passing through an element is a parallelogram.*



GIVEN—the cylinder RS of which $ABDC$ is a section made by a plane passing through an element AB.

TO PROVE $ABDC$ is a parallelogram.

First, the lines AC and BD are straight and parallel. §§ 528, 544

Since BA is an element, and therefore straight, we have only to prove that DC is straight and is parallel to BA.

Through D draw a straight line parallel to BA.

This line will lie in the cylindrical surface, by definition.

It will also lie in the plane determined by BA and D. § 526 II, IV

It therefore coincides with DC.

Hence DC is straight and is parallel to BA.

Therefore $ABDC$ is a parallelogram. § 114

Q. E. D.

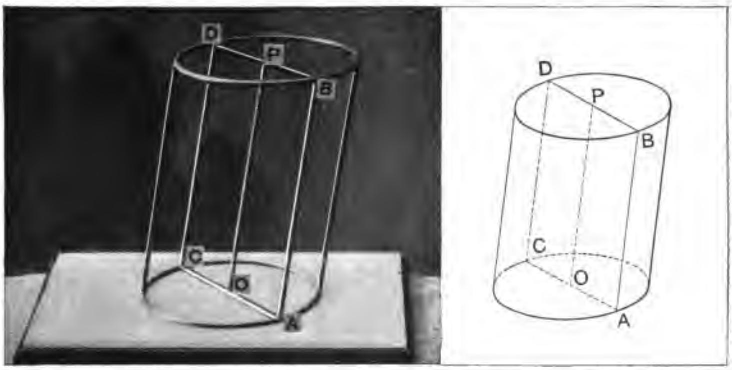
772. *Def.*—A **right cylinder** is one whose elements are perpendicular to its bases.

773. COR. *Every section of a right cylinder made by a plane perpendicular to its base is a rectangle.*

774. *Defs.*—A **circular cylinder** is one whose bases are circles. The straight line joining the centres of its bases is called the **axis** of the circular cylinder.

PROPOSITION III. THEOREM

775. *The axis of a circular cylinder is equal and parallel to its elements.*



GIVEN a circular cylinder AD, whose axis is OP.

TO PROVE—OP is equal and parallel to any element AB.

Draw through B and P the diameter BD of the upper base, and let CD be the element passing through D.

Then pass a plane through AB and CD cutting the lower base in AC.

We have AC parallel to BD. § 544

Hence $AC=BD$. § 118

Therefore AC passes through O and is a diameter of the lower base. § 170

Hence $AO = BP$. § 158

Also AO was proved parallel to BP.

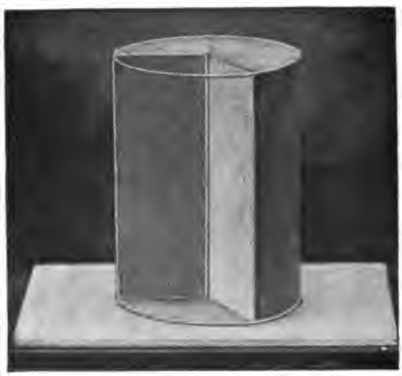
Hence the figure $ABPO$ is a parallelogram. § 126

Therefore OP is equal and parallel to AB. § 117

Q. E. D.

776. COR. I. *The axis of a circular cylinder passes through the centres of all sections parallel to its base.*

777. COR. II. *A right circular cylinder may be generated by the revolution of a rectangle about one of its sides as an axis.*



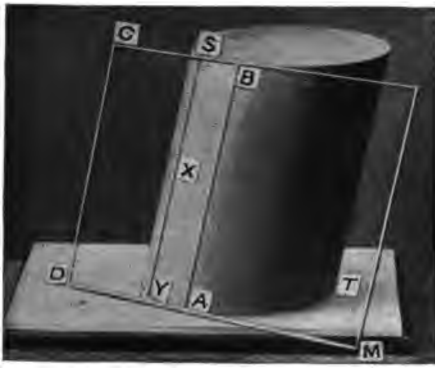
778. *Defs.*—For this reason a right circular cylinder is also called a **cylinder of revolution.**

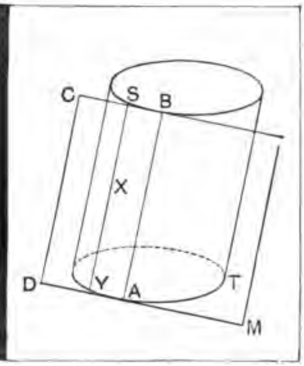
The radius of the base of a cylinder of revolution is called the **radius of the cylinder.**

779. *Def.*—A plane is **tangent** to a cylinder when it passes through an element and meets its surface nowhere else.

PROPOSITION IV. THEOREM

780. *A plane passing through a tangent to the base of a cylinder and the element drawn at the point of contact is tangent to the cylinder.*





GIVEN—the cylinder ST, the tangent AD to its base, and the element AB drawn through the point of contact.

TO PROVE—that the plane CM, passing through AD and AB, is tangent to the cylinder.

If the plane should meet the surface of the cylinder in any point X, not in AB, draw the element SY passing through X.

Then SY would lie in the plane CM. § 526 II, IV

Therefore AD would meet the curve AT in two points, A and Y.

This cannot be, since AD is tangent to the base.

Hence the plane CM does not meet the surface of the cylinder except in AB.

It is therefore tangent to the cylinder. § 779

Q. E. D.

781. COR. I. *Through a given element, one and only one plane tangent to the cylinder can be drawn.*

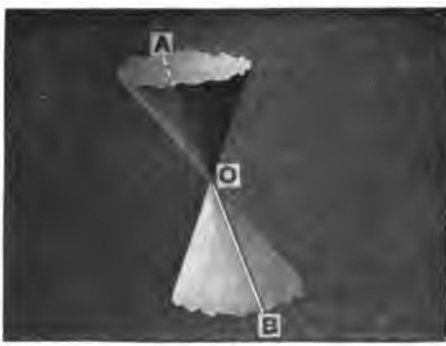
782. COR. II. *If a plane is tangent to a cylinder, its intersection with the plane of the base is tangent to the base.*

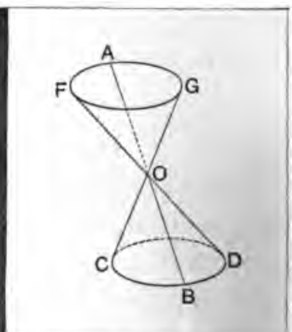
783. COR. III. *The intersection of two planes tangent to a cylinder is parallel to the elements.*

784. *Exercise.*—Show how to draw a plane through a given point tangent to a cylinder.

THE CONE

785. *Def.*—A **conical surface** is a surface generated by a moving straight line which continually intersects a given fixed curve and constantly passes through a given fixed point.





Thus, if the straight line OB passes through the point O and moves so as continually to intersect the curve CD, the surface generated $O-CBD$ is a conical surface.

786. *Defs.*—The moving line is called the **generatrix**; the fixed curve the **directrix**; the fixed point the **vertex.**

Any straight line in the surface, as OA, representing one position of the generatrix, is called an **element** of the surface.

787. *Remark.*—If the generatrix is of indefinite length, as BOA, the conical surface consists of two symmetrical parts, each of indefinite extent, lying on opposite sides of the vertex, as $O-CBD$ and $O-GAF$.

The directrix may be any curve whatever. But for the student who has not studied the appendix the proofs are rigorous only when the directrix is considered to be the circumference of a circle.

788. *Defs.*—A **cone** is a solid bounded by a closed conical surface and a plane.

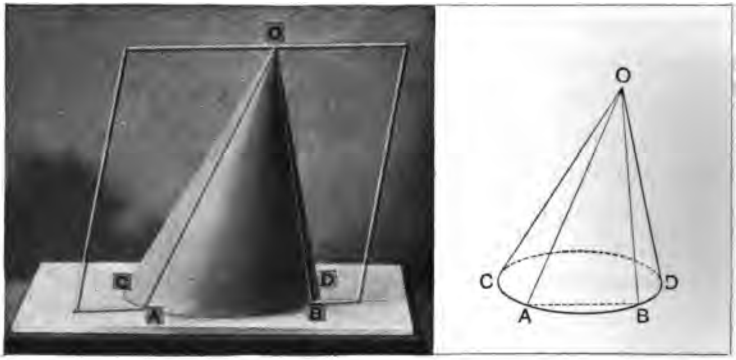
The conical surface is called the **lateral surface** and the section made by the plane the **base** of the cone.

The vertex of the conical surface is called the **vertex** of the cone, and the elements of the conical surface are also called **elements of the cone.**

CONES

PROPOSITION V. THEOREM

789. *Every section of a cone made by a plane passing through its vertex and cutting its base is a triangle.*



GIVEN—the cone $O—CABD$, whose base is cut in the line AB by a plane passed through O.

TO PROVE—the section made by this plane is a triangle.

The intersection AB is a straight line. § 528

We therefore need only to prove that the intersections OA and OB are straight.

Draw straight lines from O to A and B.

These straight lines lie in both the cutting plane and the conical surface. §§ 524, 785

Therefore they form the intersections of this plane and the conical surface.

Hence the section made by the plane OAB is a triangle.
Q. E. D.

790. *Defs.*—A cone whose base is a circle is called a **circular cone.** The straight line joining the vertex of a circular cone to the centre of its base is the **axis** of the cone.

791. *Every section o*
allel to its base is a circ
the axis with the plane p

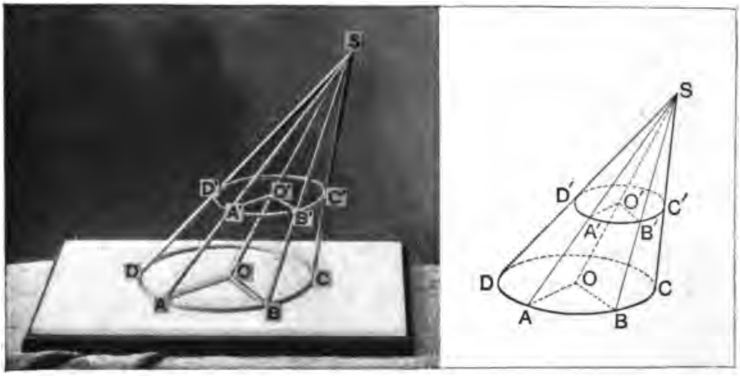
GIVEN—the circular cone S
made by a plane parallel t
Let the axis SO intersec

TO PROVE—that $A'B'C'D'$ is

Let A' and B' be an
$A'B'C'D'$.

Pass a plane through S
and B'.

Let SA and SB be th
intersect the conical surf
the straight lines in which
Then $A'O'$ is parallel to



Therefore the triangle SOA is similar to $SO'A'$, and SOB to $SO'B'$. § 275

Therefore $\dfrac{A'O'}{AO}=\dfrac{SO'}{SO}$ and $\dfrac{B'O'}{BO}=\dfrac{SO'}{SO}$. § 274

Hence $\dfrac{A'O'}{AO}=\dfrac{B'O'}{BO}$.

But $AO=BO$. § 150

Therefore $A'O'=B'O'$.

Since A' and B' were taken as *any* two points in the perimeter of the section, all points in this perimeter are equidistant from O'.

Therefore $A'B'C'D'$ is a circle, and its centre is O'. Q. E. D.

792. *Def.*—A **right circular cone** is a circular cone whose axis is perpendicular to its base.

PROPOSITION VII. THEOREM

793. *A right circular cone may be generated by the revolution of a right triangle about one of its sides as an axis.*

The proof is left to the student.

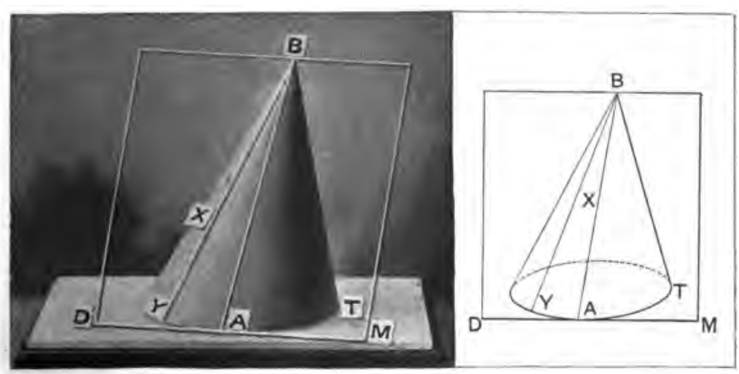
794. *Def.*—From its mode of generation a right circular cone is also called a **cone of revolution.**

795. COR. *The elements of a cone of revolution are all equal.*

796. *Def.*—A plane is **tangent** to a cone when it passes through an element and meets its surface in no other point.

PROPOSITION VIII. THEOREM

797. *A plane passing through a tangent to the base of a cone and the element drawn to the point of contact is tangent to the cone.*



GIVEN—the cone ABT, the tangent AD to its base, and the element AB drawn through the point of contact.

TO PROVE—that the plane BM, passing through AD and AB, is tangent to the cone.

If this plane should meet the surface of the cone in any point X, not in AB, draw the element BY passing through X.

Then BY would lie in the plane BM. § 524

Therefore AD would meet the curve AT in two points, A and Y.

This is contrary to the hypothesis that AD is tangent to the base.

Hence the plane BM does not meet the surface of the cone except in AB.

It is therefore tangent to the cone. § 796

Q. E. D.

798. Cor. I. *Through a given element, one and only one plane tangent to the cone can be drawn.*

799. Cor. II. *If a plane is tangent to a cone, its intersection with the plane of the base is tangent to the base.*

800. *Exercise.*—Show how to draw a plane through a given point tangent to a cone.

THE SPHERE

801. *Defs.*—A **spherical surface** is a closed surface all points of which are equidistant from a point called the **centre.**

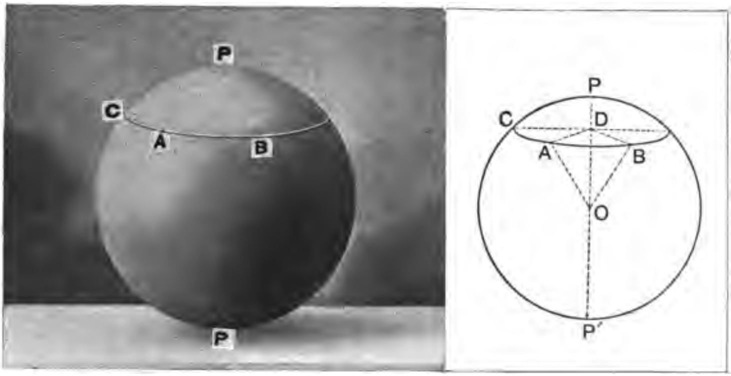
802. *Defs.*—A **sphere** is a solid bounded by a spherical surface.

A **radius** of the sphere is a straight line joining the centre to a point of the surface.

A **diameter** of the sphere is a straight line drawn through the centre and terminated at both ends by the surface.

PROPOSITION IX. THEOREM

803. *Every section of a sphere made by a plane is a circle whose centre is the foot of the perpendicular from the centre of the sphere on that plane.*



GIVEN—the sphere whose centre is O, cut by a plane in the section CAB.

Draw OD perpendicular to the cutting plane, meeting it at D.

TO PROVE—that CAB is a circle and that D is its centre.

Let A and B be any two points in the perimeter of CAB.

Join AD and BD.

Now $OA = OB$. § 801

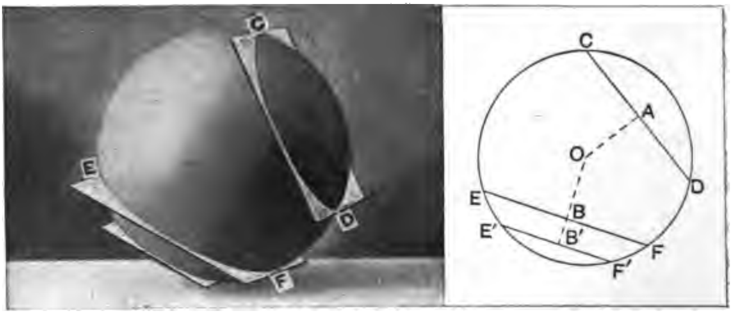
Therefore $DA = DB$. § 540 I

Since A and B are any two points in the perimeter of CAB, all points in this perimeter are equidistant from D.

Therefore CAB is a circle and D is its centre. Q. E. D.

804. COR. I. *If a plane is passed through the centre of a sphere, the centre of the circle thus formed is the centre of the sphere, and its radius is the radius of the sphere.*

805. COR. II. *Circles of the sphere equidistant from its centre are equal; and conversely.*



Hint.—This is proved by dropping perpendiculars OA and OB from the centre of the sphere on the planes of the two circles.

We then pass a plane through OA and OB intersecting the sphere in the circle $CDFE$ and the two circles in question in the diameters CD and EF.

The proof then consists in applying §§ 170, 158.

806. COR. III. *The more distant a circle of the sphere is from its centre, the smaller is the circle; and conversely.*

807. *Def.*—A circle whose plane passes through the centre of the sphere is called a **great circle.**

808. *Def.*—A circle whose plane does not pass through the centre of the sphere is called a **small circle.**

809. COR. IV. *All great circles are equal.*

810. COR. V. *A*

Hint.—Since they h

of each circle.

811. COR. VI.

its surface into two

Hint.—Prove by su

812. COR. VII.

of a sphere one and

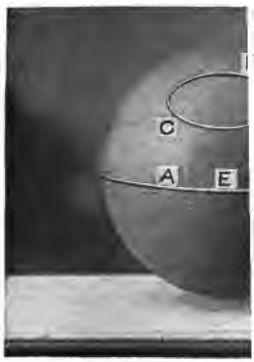
813. COR. VIII.

of a sphere, not at t

one great circle can

Hint.—The two poi

the plane of a great cir



Question.—If the two

Corollary VIII. modifie

814. *Def.*—By th

surface of a sphere is

less than a semi-circ

Thus the distance bet

815. *Def.*—The

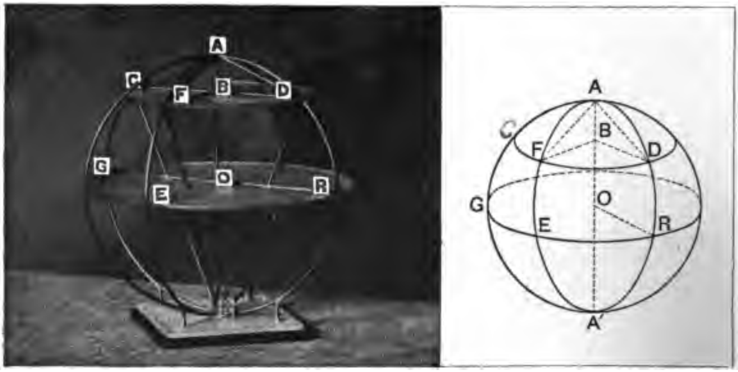
dicular to the plane

axis of that circle.

816. *Def.*—The **poles** of a circle are the extremities of its axis.

PROPOSITION X. THEOREM

817. *All points in the circumference of a circle of the sphere are equally distant from each of its poles.*



GIVEN—any two points F and D in the circumference of a circle CFD and A and A', the poles of CFD.

Draw the great-circle arcs AF, AD, $A'F$, $A'D$.

TO PROVE arc AF=arc AD, and arc $A'F$=arc $A'D$.

Let B be the intersection of the axis AA' with the plane of CFD. Draw the straight lines AF and AD.

Now $BF=BD$. § 803

Hence chord AF=chord AD. § 539 I

Therefore arc AF=arc AD. § 164

Similarly we may prove

arc $A'F$=arc $A'D$. Q. E. D.

818. *Def.*—The **polar distance** of a circle of a sphere is the arc of a great circle drawn from its nearest pole to any point of its circumference.

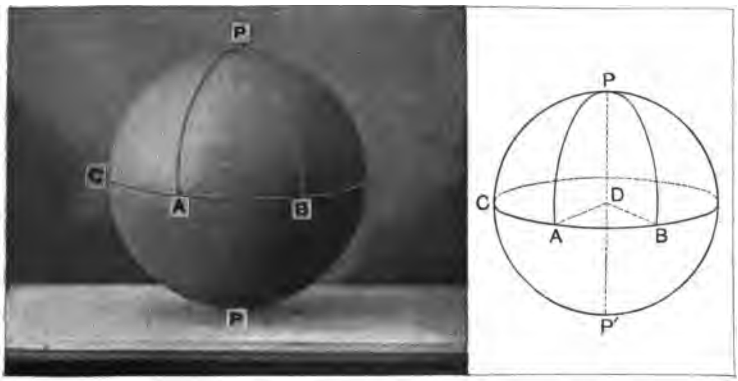
819. Cor. *The polar distance of a great circle is a quadrant of a great circle.*

Hint.—Let GER be a great circle. Then its centre O is also the centre of the great circle ARA'. Hence the arc AR measures the right angle AOR.

820. *Def.*—The term **quadrant** in connection with a sphere is used to signify a quadrant of a great circle.

PROPOSITION XI. THEOREM

821. *If a point on the surface of a sphere is at a quadrant's distance from two points on that surface, it is the pole of the great circle passed through those points.*



Given—a point P on the surface of a sphere at a quadrant's distance from each of the points A and B on that surface.

To prove that P is the pole of the great circle AB.

Draw the radii DP, DA, and DB.

Since PA and PB are quadrants, PDA and PDB are right angles. §§ 804, 194

Therefore PD is perpendicular to the plane DAB. § 531

That is, P is the pole of the great circle AB. § 816

Q. E. D.

822. *Remark.*—The preceding theorems enable us to draw circumferences upon the surface of a sphere as easily as upon a plane. A pair of compasses with curved branches is employed. The opening of the compasses (distance between their points) is made equal to the chord of the polar distance of the required circle. Then, one point of the compasses being placed at the pole, the other describes the circumference.

If we wish to draw an arc of a *great* circle, the opening of the compasses must be equal to the chord of a quadrant. This can be found when the diameter is known. A method for finding the diameter will be given in the next proposition.

If it is desired to draw an arc of a great circle *through two points* on the surface, it is necessary first to find the pole of this circle. For this purpose draw circumferences of great circles with the two points as poles. These two circumferences will intersect in two points, either of which is the required pole. Then the circumference can be drawn as described above.

PROPOSITION XII. PROBLEM

823. *To find the diameter of a given sphere.*

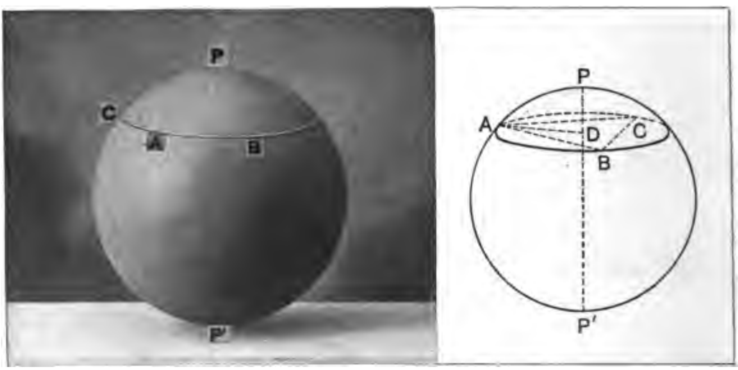


FIG. 1

We suppose the given sphere a material one, and that only measurements on its surface are possible.

First, with any point P on the surface as a pole, and with any opening of the compasses AP, draw a circumference ABC on the surface (Fig. 1). Then the straight line AP is known.

Take any three points A, B, C in this circumference. Measure with the compasses the straight lines AB, BC, CA.

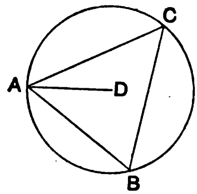


FIG. 2

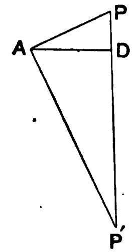


FIG. 3

Secondly, on a plane construct a triangle having AB, BC, CA as sides (Fig. 2). § 90

Find the centre D of the circle circumscribing ABC. § 219

Then the straight line AD is known.

Thirdly, with AD as a side and AP as the hypotenuse, construct the right triangle ADP (Fig. 3).

Draw AP' perpendicular to AP, meeting PD produced in P'.

Then PP' is equal to the diameter of the given sphere.

Proof.—In Fig. 1 draw PP' the axis of the circle ABC meeting the plane of ABC in D. Then D is the centre of the circle ABC. § 803

Draw DA and $P'A$.

The triangle ABC (Fig. 1) equals the triangle ABC (Fig. 2). § 89

Hence AD is the same in Figs. 1, 2, and 3. § 158

Now in Fig. 1 the angle PDA is right. § 530

And AP is the same in Figs. 1 and 3. Cons.

Hence the right triangles ADP are equal in Figs. 1 and 3. § 101

Again in Fig. 1 the angle PAP' is right. § 202

Hence the right triangles PAP' are equal in Figs. 1 and 3. § 86

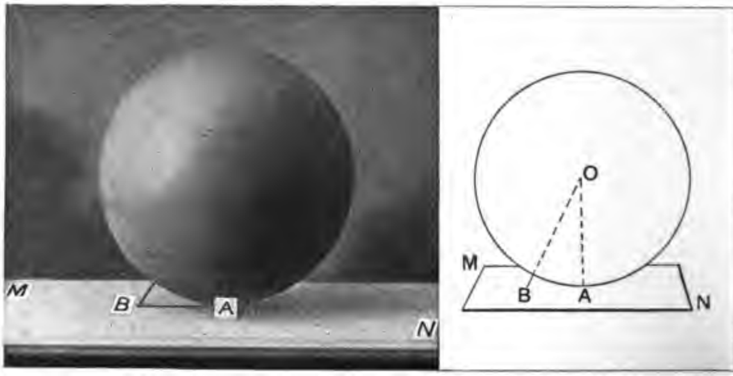
Therefore PP' in Fig. 3 is equal to the diameter of the given sphere. Q. E. F.

824. *Defs*.—A plane is **tangent** to a sphere when it has one, and only one, point in common with the surface of the sphere. This point is called the **point of tangency.**

In the same case the sphere is said to be tangent to the plane.

PROPOSITION XIII. THEOREM

825. *A plane perpendicular to a radius of a sphere at its extremity is tangent to the sphere; conversely, a plane tangent to a sphere is perpendicular to the radius drawn to the point of tangency.*



GIVEN—the plane MN perpendicular to the radius OA of the sphere whose centre is O at its extremity A.

TO PROVE that MN is tangent to the sphere.

Let B be *any* point in MN other than A. Join OB.

Then $OB > OA$. § 536

Hence B is outside the sphere. § 801

That is, MN has only one point A in common with the surface of the sphere.

Therefore MN is tangent to the sphere. § 824

Q. E. D.

CONVERSELY:

GIVEN—the plane MN tangent to the sphere whose centre is O. Draw the radius OA to the point of tangency.

TO PROVE that MN is perpendicular to OA.

Let B be any point in MN other than A. Join OB.

Since MN is tangent to the sphere at A, B lies outside of the sphere. § 824

Hence $OB > OA$.

That is, OA is the shortest line from O to MN.

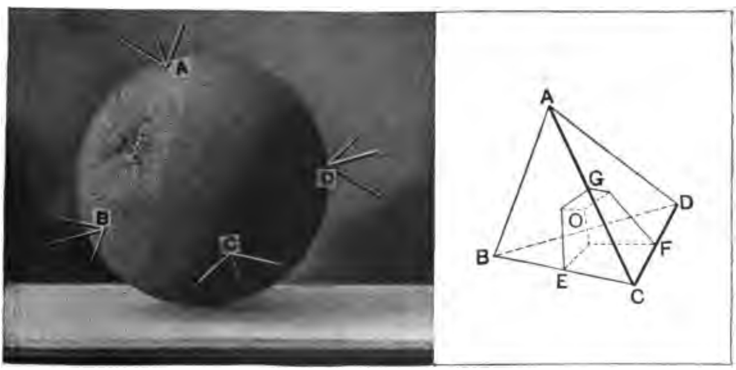
Therefore MN is perpendicular to OA. § 536

Q. E. D.

826. *Exercise.*—Prove that three planes perpendicular respectively to the three edges of a triedral angle meet in a point.

PROPOSITION XIV. THEOREM

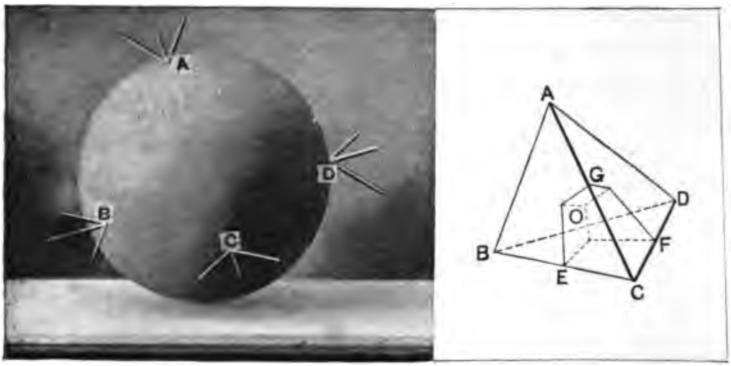
827. *A spherical surface can be passed through any four points, not in the same plane, and but one.*



GIVEN the four points A, B, C, D, not in the same plane.

TO PROVE—that one, and only one, spherical surface can be passed through these points.

Form a tetraedron having these points as vertices.



Draw three planes EO, FO, and GO perpendicular respectively to the edges BC, CD, and CA at their middle points.

The plane EO is the locus of points equidistant from B and C; the plane FO is the locus of points equidistant from C and D; and the plane GO is the locus of points equidistant from C and A. § 611

Hence the intersection O of these three planes is equidistant from A, B, C, and D, and is the only point equidistant from those points. § 102

Therefore the spherical surface described with O as a centre, and the line OA as a radius, will pass through the four points, and will be the only spherical surface that can be passed through the four points. Q. E. D.

828. COR. I. *The six planes perpendicular to the six edges of a tetraedron at their middle points meet in a point.*

829. COR. II. *The four straight lines perpendicular to the faces of a tetraedron at the centres of their circumscribing circles meet in a point.*

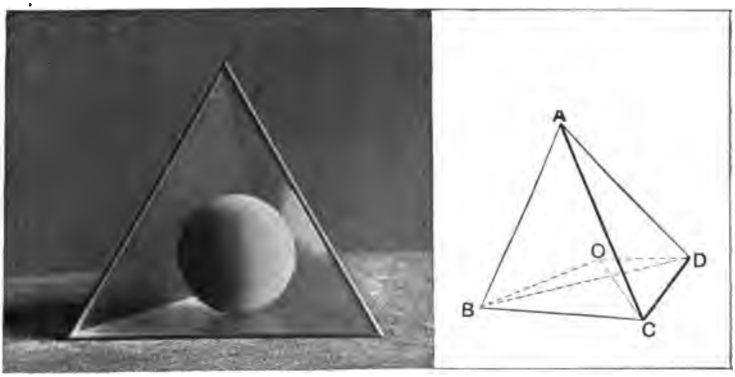
830. *Exercise.*—Prove that the three planes bisecting

the diedral angles at the base of a tetraedron meet in a point.

831. *Def.*—A sphere is **inscribed** in a polyedron when its centre is within the polyedron and its surface is tangent to all the faces of the polyedron.

PROPOSITION XV. THEOREM

832. ***A sphere can be inscribed in any tetraedron, and but one.***



GIVEN the tetraedron *ABCD*.

TO PROVE—that one, and only one, sphere can be inscribed in it.

Bisect the diedral angles *BC*, *CD*, and *DB* by the planes *BOC*, *COD*, and *DOB*.

The plane *BOC* is the locus of points equidistant from the faces *BCD* and *BAC*; the plane *COD* is the locus of points equidistant from the faces *BCD* and *CAD*; and the plane *DOB* is the locus of points equidistant from the faces *BCD* and *DAB*. § 580

Hence the intersection O of these three planes is equidistant from the four faces of the tetraedron, and is the only point equidistant from the faces.

Therefore the sphere described with O as a centre, and the perpendicular distance from O upon the face BCD as a radius, will be tangent to all the faces of the tetraedron and hence will be inscribed in the tetraedron. § 825

And it will be the only sphere that can be inscribed in the tetraedron. Q. E. D.

833. COR. *The six planes bisecting the six diedral angles of a tetraedron meet in a point.*

SPHERICAL ANGLES

834. *Def.*—The **angle of two curves** meeting in a common point is the angle formed by the two tangents to the curves at that point.

835. *Def.*—A **spherical angle** is the angle between two intersecting arcs of great circles on the surface of a sphere.

PROPOSITION XVI. THEOREM

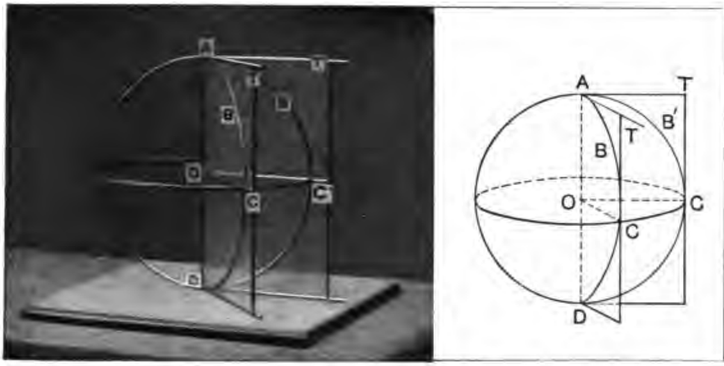
836. *The angle of two arcs of great circles on a spherical surface is*

I. *Equal to the plane angle of the diedral angle formed by their planes.*

II. *Measured by the arc of a great circle described with its vertex as a pole and included between its sides, produced if necessary.*

GIVEN—AB and AB', two arcs of great circles whose planes form a diedral angle having the diameter AD for edge.

With A as a pole describe a great circle cutting AB and AB', produced, if necessary, in C and C'.



I. To prove—the angle BAB' is equal to the plane angle of the diedral angle $BADB'$.

Draw AT and AT' tangent to the arcs AB and AB' respectively.

Then by definition the angles BAB' and TAT' are identical. § 834

But AT and AT' are perpendicular to OA. § 173

Hence TAT', or BAB', is the plane angle of the diedral angle $BADB'$. § 567

Q. E. D.

II. To prove—that the angle BAB' is measured by the arc CC'.

Join the centre of the sphere, O, to C and C'.

Then, since A is the pole of CC', the plane COC' is perpendicular to AO. § 816

Hence COC' is the plane angle of the diedral angle $BADB'$. § 530

Therefore the angle BAB' is equal to the angle COC'.

But COC' is measured by the arc CC'. § 191

Therefore BAB' is measured by the arc CC'. Q. E. D.

837. Cor. I. *Any great-circle-arc AC, drawn through the pole of a given great circle CC' is perpendicular to the circumference CC'.*

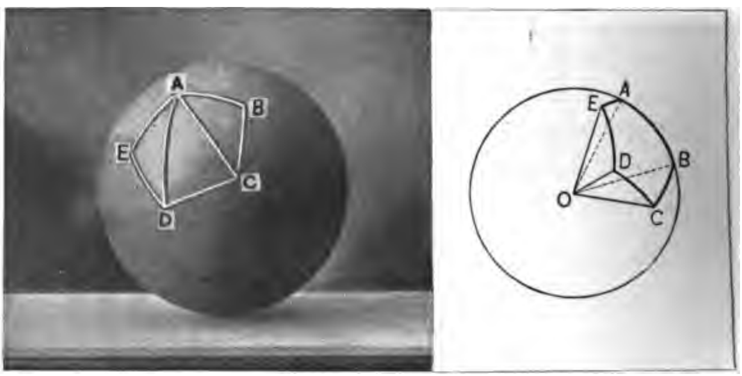
Hint.—First prove the plane AOC perpendicular to the plane COC'. Then the plane angle of the diedral angle OC is a right angle.

838. Cor. II. Conversely, *any great-circle-arc perpendicular to a given arc must pass through the pole of the given arc.*

Hint.—Apply § 576.

SPHERICAL POLYGONS

839. *Defs.*—A **spherical polygon** is a portion of a spherical surface bounded by three or more arcs of great circles; as $ABCDE$.



The bounding arcs are called the **sides** of the spherical polygon; their intersections, the **vertices**; and the angles formed by the sides at the vertices, the **angles** of the spherical polygon.

840. *Def.*—A **diagonal** of a spherical polygon is an arc of a great circle joining any two vertices not consecutive.

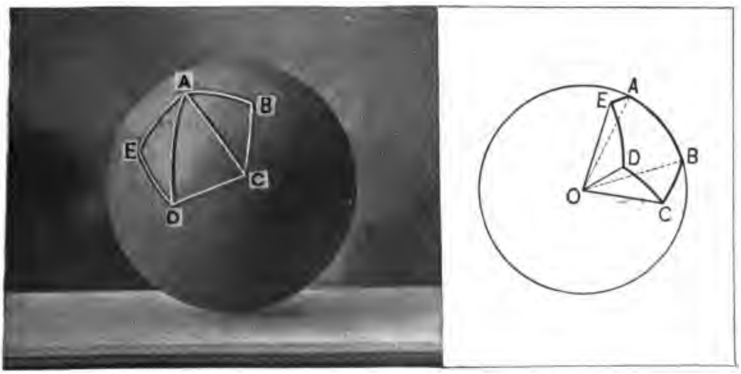
841. *Remark.*—The sides of a spherical polygon are usually measured in degrees.

842. *Def.*—The polyedral angle, whose vertex is at the centre of the sphere, formed by the planes of the sides of a spherical polygon, is said to **correspond** to the spherical polygon.

Thus the polyedral angle $O-ABCDE$ corresponds to the spherical polygon $ABCDE$.

PROPOSITION XVII. THEOREM

843. *The sides of a spherical polygon measure the corresponding face angles of the corresponding polyedral angle; and its angles are equal to the plane angles of the corresponding diedral angles.*



Hint.—This proposition is an immediate consequence of §§ 191, 836 I.

844. *Remark.*—Since each face angle of a polyedral angle is assumed to be less than two right angles, each side of a spherical polygon will be assumed to be less than a semicircumference.

845. *Def.*—The **parts** of a spherical polygon are its sides and angles.

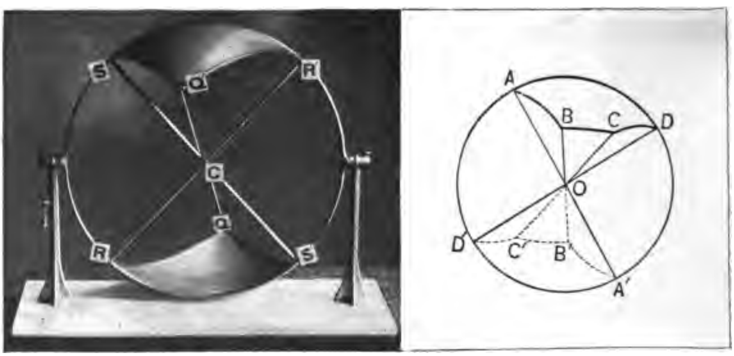
846. *Remark.*—By means of the relations between the parts of a spherical polygon and the parts of its corresponding polyedral angle we can, from any property of polyedral angles, deduce an analogous property of spherical polygons.

Reciprocally, from any property of spherical polygons, we can infer an analogous property of polyedral angles.

847. *Defs.*—A **spherical triangle** is a spherical polygon of three sides. It is called **isosceles, equilateral,** or **right-angled** in the same cases in which a plane triangle would be so named.

SYMMETRICAL SPHERICAL TRIANGLES AND POLYGONS

848. *Def.*—Two spherical polygons are **vertical** when their vertices are situated by pairs at opposite ends of the same diameter.



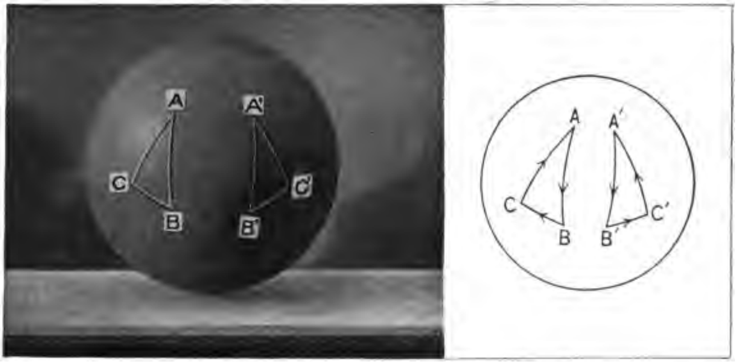
Thus, to determine the spherical polygon vertical to $ABCD$ we draw the diameters AOA', BOB', COC', DOD'. Then $A'B'C'D'$ is vertical to $ABCD$.

849. THEOREM. *Two spherical polygons are vertical, if their corresponding polyedral angles are vertical, and conversely.*

Hint.—This follows immediately from the preceding definition and § 599.

850. *Def.*—Two spherical polygons are **symmetrical** when they have the same number of parts equal each to each and arranged in opposite order.

Thus, in the triangles ABC and $A'B'C'$, if $A=A'$, $B=B'$, $C=C'$, $AB=A'B'$, $BC=B'C'$, $CA=C'A'$, and the order of arrangement of the parts is opposite in the two figures, the triangles are symmetrical.



The meaning of the words "arranged in opposite order" will be made clearer by the following explanation:

In the figure above the direction of motion in going from A to B to C to A is the direction of rotation of the hands of a clock; the direction of motion in going from A' to B' to C' to A' is opposite to the direction of rotation of the hands of a clock; supposing that in each case we look at the surface of the sphere from the outside. If we look at the surface from the inside, the directions will be reversed.

851. THEOREM. *Two spherical polygons are symmetrical, if their corresponding polyedral angles are symmetrical, and conversely.*

This follows immediately from the preceding definition and §§ 600, 843.

PROPOSITION XVIII. THEOREM

852. *Two vertical spherical polygons are symmetrical.*

Proof.—The corresponding polyedral angles at the centre are vertical. § 849

They are therefore symmetrical. § 601

Hence the spherical polygons are symmetrical. § 851

Q. E. D.

PROPOSITION XIX. THEOREM

853. *Of two symmetrical spherical polygons either is equal to the vertical of the other.*

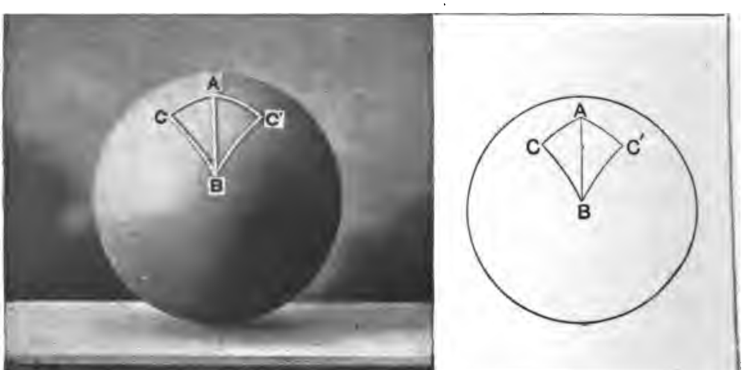
Proof.—The corresponding polyedral angles at the centre are symmetrical. § 851

Hence either may be made to coincide with the vertical of the other. § 602

When this is done, the two spherical polygons will be vertically opposite. § 849

Q. E. D.

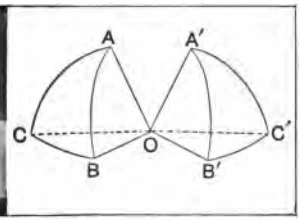
854. *Remark.*—In general two symmetrical spherical polygons cannot be made to coincide, and hence are not equal.



Thus, if two symmetrical spherical triangles ABC and $A'B'C'$ are not isosceles, the only side of $A'B'C'$ with which AB can be made to coincide is $A'B'$. If we place A upon A' and B upon B', C and C' will fall on opposite sides of AB. If we place A upon B' and B upon A', C and C' will fall on the same side of AB, but will not coincide. But if the triangles are *isosceles*, they can be made to coincide, as the following proposition will show.

PROPOSITION XX. THEOREM

855. *Two symmetrical isosceles spherical triangles are equal.*



Hint.—Show that the corresponding triedral angles have two face angles and the included diedral angle respectively equal, and similarly arranged (A corresponding to A', but B to C' and C to B').
Then superpose these triedral angles (§ 595).

856. COR. I. *In an isosceles spherical triangle the angles opposite the equal sides are equal.*

Hint.—In superposing the symmetrical isosceles triangles in the above figure, the angle B' is made to coincide with C. But we know that $B'=B$.

857. COR. II. *If a spherical triangle is equilateral, it is also equiangular.*

858. COR. III. *If two face angles of a triedral angle are equal, the opposite diedral angles are equal.*

859. COR. IV. *If the three face angles of a triedral angle are equal, its three diedral angles are equal.*

PROPOSITION XXI. THEOREM

860. *If two angles of a spherical triangle are equal, the opposite sides are equal.*

Hint.—Form the symmetrical triangle. Show that the corresponding triedral angles have a face angle and the adjacent diedral angles respectively equal, and similarly arranged. Then superpose these triedral angles (§ 596).

861. COR. I. *If a spherical triangle is equiangular, it is also equilateral.*

862. COR. II. *If two diedral angles of a triedral angle are equal, the opposite face angles are equal.*

863. COR. III. *If the three diedral angles of a triedral angle are equal, the three face angles are equal.*

PROPOSITION XXII. THEOREM

864. *Any side of a spherical triangle is less than the sum of the two others.*

Hint.—Form the corresponding triedral angle.

Then apply §§ 843, 593.

865. COR. I. *Any side of a spherical polygon is less than the sum of all the others.*

Hint.—Divide the polygon into triangles by diagonals from any vertex.

866. COR. II. *Any face angle of a polyedral angle is less than the sum of all the others.*

867. *Def.*—A spherical polygon is **convex** when its corresponding polyedral angle is convex.

PROPOSITION XXIII. THEOREM

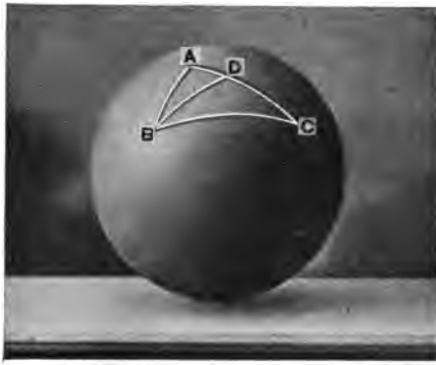
868. *The sum of the sides of a convex spherical polygon is less than the circumference of a great circle.*

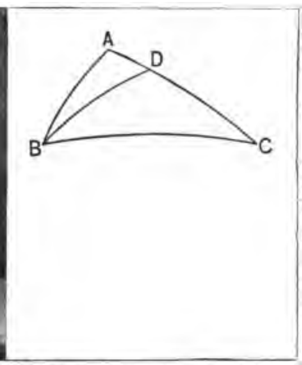
Hint.—Form the corresponding polyedral angle.

Then apply §§ 843, 594.

PROPOSITION XXIV. THEOREM

869. *If two angles of a spherical triangle are unequal, the opposite sides are unequal, and the greater side is opposite the greater angle.*





GIVEN the spherical triangle ABC in which angle $ABC > ACB$.

TO PROVE side $AC > AB$.

Draw BD making angle $DBC = DCB$.

Then $DC = DB$. § 860

Adding AD to each of these equals we have

$$AC = AD + DB.$$

But $AD + DB > AB$. § 864

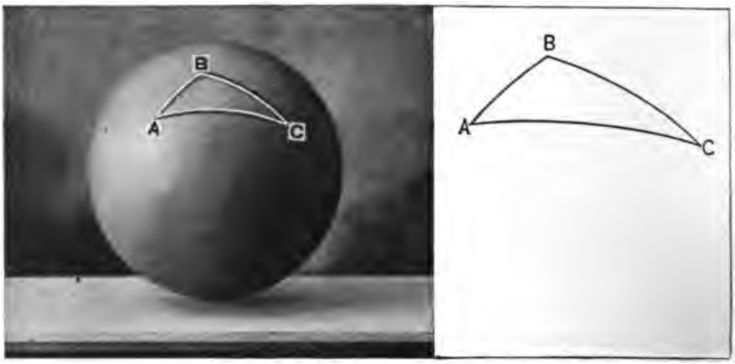
Therefore $AC > AB$. Q. E. D.

870. COR. *If two diedral angles of a triedral angle are unequal, the opposite face angles are unequal, and the greater face angle is opposite the greater diedral angle.*

PROPOSITION XXV. THEOREM

871. *If two sides of a spherical triangle are unequal, the opposite angles are unequal, and the greater angle is opposite the greater side.*

[Converse of Proposition XXIV.]



GIVEN the spherical triangle ABC in which side $AC > AB$.

TO PROVE angle $ABC > ACB$.

If ABC were equal to ACB, then AC would equal AB. § 860

If ABC were less than ACB, then AC would be less than AB. § 869

Both these conclusions are contrary to the hypothesis.

Therefore $ABC > ACB$. Q. E. D.

872. COR. *If two face angles of a triedral angle are unequal, the opposite diedral angles are equal, and the greater diedral angle is opposite the greater face angle.*

PROPOSITION XXVI. THEOREM

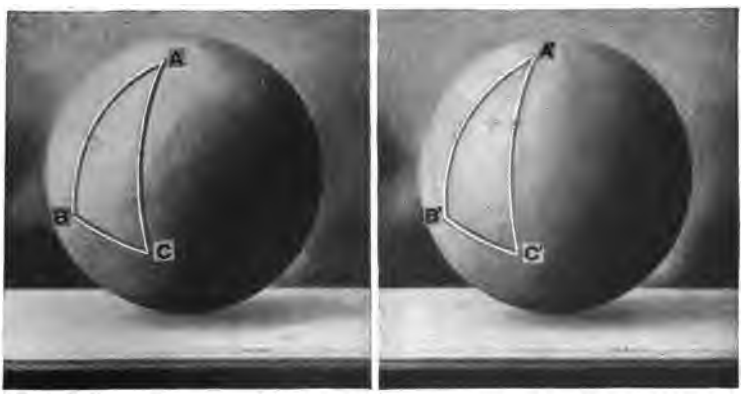
873. *Two triangles on the same sphere are equal:*

I. *If two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*

II. *If a side and the two adjacent angles of one are equal respectively to a side and the two adjacent angles of the other.*

III. *If the three sides of one are equal respectively to the three sides of the other.*

Provided in each case that the parts given equal are arranged in the same order in both triangles.



Proof.—In each case the corresponding triedral angles are equal. §§ 595, 596, 597

They can therefore be placed in coincidence.

At the same time the triangles coincide.

Therefore the two given triangles are equal. Q. E. D.

PROPOSITION XXVII. THEOREM

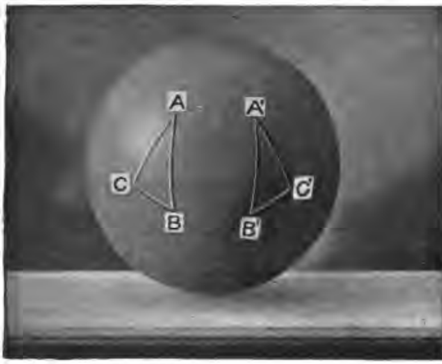
874. *Two triangles on the same sphere are symmetrical:*

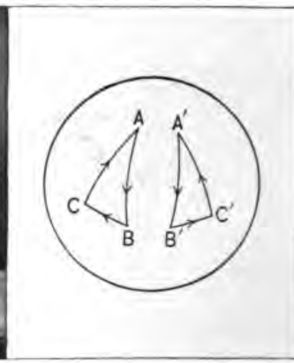
I. *If two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*

II. *If a side and the two adjacent angles of one are equal respectively to a side and the two adjacent angles of the other.*

III. *If the three sides of one are equal respectively to the three sides of the other.*

Provided in each case that the parts given equal are arranged in opposite order in the two triangles.





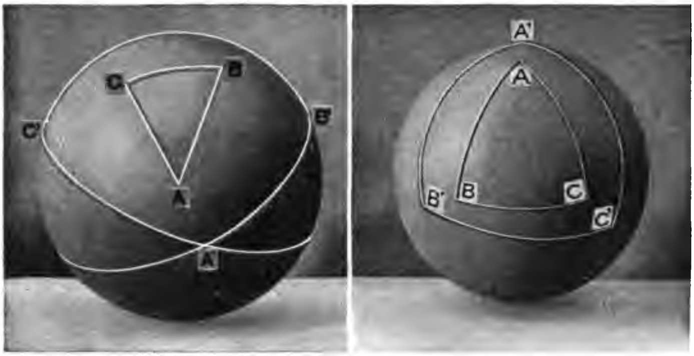
Proof.—In each case the corresponding triedral angles at the centre are symmetrical. § 603

Therefore the two given triangles are symmetrical. § 851

Q. E. D.

POLAR TRIANGLES

875. *Def.*—If, with the vertices of a spherical triangle as poles, arcs of great circles are described, these arcs will divide the spherical surface into eight triangles. One of these is called the **polar triangle** of the given triangle.



The method of selecting the polar triangle from the eight is as follows: Call the given triangle ABC and the polar triangle $A'B'C'$. Then A' is one of the intersections of the arcs described from B and C as poles; that one which is less than a quadrant's distance from A. In a similar way B' and C' are determined.

PROPOSITION XXVIII. THEOREM

876. *If one spherical triangle is the polar triangle of another, then, reciprocally, the second spherical triangle is the polar triangle of the first.*

GIVEN that $A'B'C'$ is the polar triangle of ABC.

TO PROVE that ABC is the polar triangle of $A'B'C'$.

Since B is the pole of $A'C'$, the distance $A'B$ is a quadrant; since C is the pole of $A'B'$, the distance $A'C$ is a quadrant. § 819

Therefore A' is the pole of BC. § 821

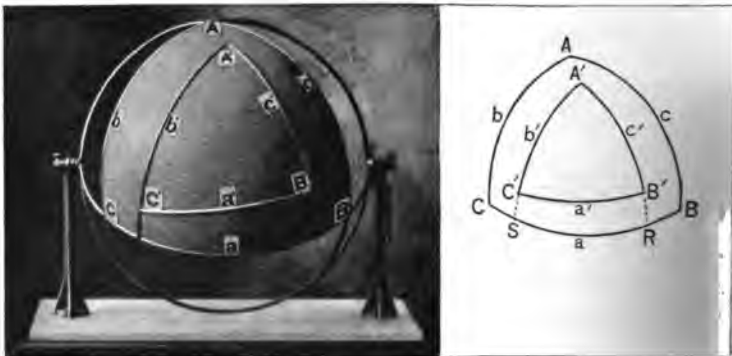
Similarly, B' is the pole of CA, and C' is the pole of AB.

Since also the distances AA', BB', and CC' are each less than a quadrant, ABC is the polar triangle of $A'B'C'$. § 875

Q. E. D.

PROPOSITION XXIX. THEOREM

877. *In two polar triangles, each angle of one is measured by the supplement of the side of which its vertex is the pole in the other.*



GIVEN—the polar triangles ABC and $A'B'C'$. Let A, B, C, and A', B', C' denote their angles, measured in degrees, and a, b, c, and a', b', c' the sides respectively opposite these angles, also measured in degrees.

TO PROVE—$A' + a = 180°$, $B' + b = 180°$, $C' + c = 180°$,
$A + a' = 18\text{c}°$, $B + b' = 180°$, $C + c' = 180°$.

Produce $A'B'$ and $A'C'$ to meet BC at R and S.

Then, since B is the pole of $A'S$ and C the pole of $A'R$, BS and CR are quadrants. § 819

Therefore $$BS + CR = 180°,$$

or $$BR + RS + RS + SC = 180°,$$

or $$RS + BC = 180°.$$

But $BC = a$, and RS measures the angle A'. § 836 II

Therefore $$A' + a = 180°.$$

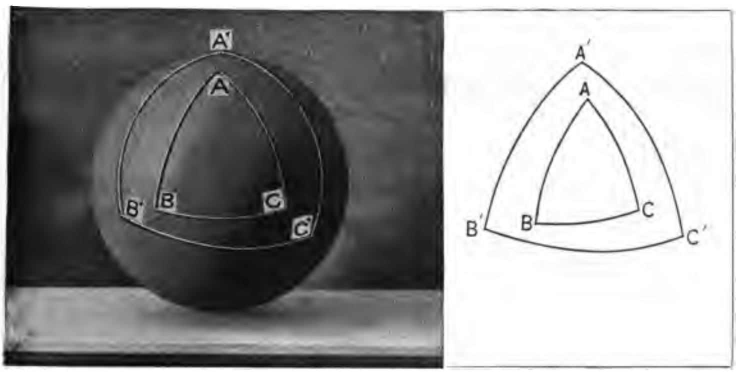
To prove the relation $A+a'=180°$ we would produce $B'C'$ to meet AB and AC.

In a similar manner the remaining relations are proved.

Q. E. D.

PROPOSITION XXX. THEOREM

878. *The sum of the angles of a spherical triangle is greater than two, and less than six, right angles.*



GIVEN the spherical triangle ABC.

Denote its angles by A, B, C, and the sides opposite in the polar triangle by a', b', c'.

TO PROVE $A+B+C>180°$ and $<540°$.

We have
$$A=180°-a'$$
$$B=180°-b'$$
$$C=180°-c'. \qquad \text{§ 877}$$

Adding these equations we get
$$A+B+C=540°-(a'+b'+c').$$

Hence $A+B+C<540°$. Q. E. D.

Also, since $a'+b'+c'<360°$, § 868

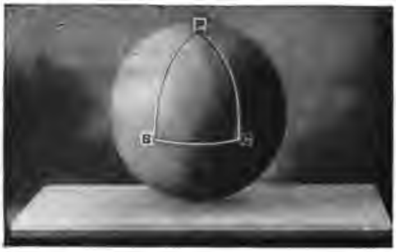
$A+B+C>180°$. Q. E. D.

879. COR. I. *A spherical triangle may have two, or even three, right angles; also two, or even three, obtuse angles.*

880. *Defs.*—A spherical triangle having two right angles is called a **bi-rectangular triangle.**

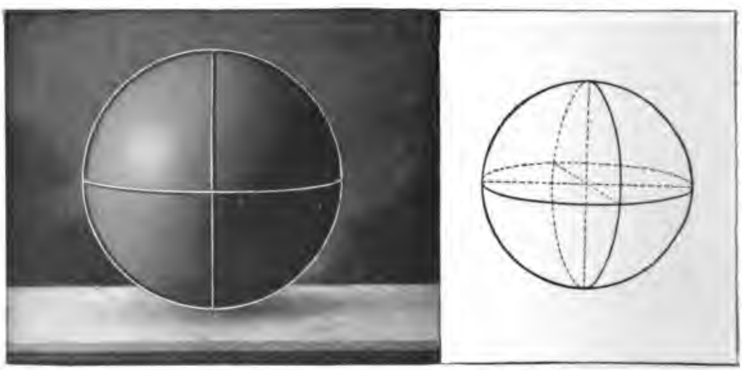
A spherical triangle having three right angles is called a **tri-rectangular triangle.**

881. COR. II. *In a bi-rectangular triangle the sides opposite the right angles are quadrants.*



Hint.—Apply §§ 838, 819.

882. COR. III. *Three planes passed through the centre of a sphere, each perpendicular to the other two, divide the surface of the sphere into eight equal tri-rectangular triangles.*

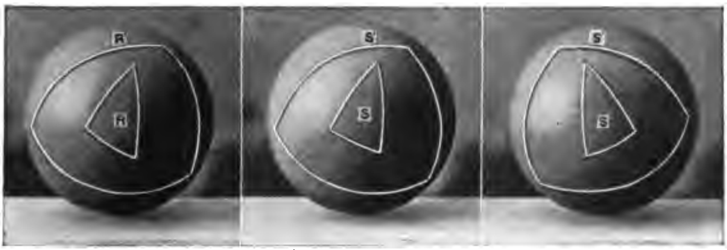


PROPOSITION XXXI. THEOREM

883. *If two triangles on the same sphere are mutually equiangular:*

I. *They are equal, when the equal angles are arranged in the same order in both triangles.*

II. *They are symmetrical, when the equal angles are arranged in opposite order in the two triangles.*



GIVEN—two mutually equiangular spherical triangles R and S.

TO PROVE—that R and S are either equal or symmetrical.

Let R' and S' be the polar triangles of R and S respectively.

Then, since R and S are mutually equiangular, we can show by means of the relations proved in Proposition XXIX. that R' and S' are mutually equilateral.

Hence R' and S' are either equal or symmetrical.

§§ 873 III, 874 III

They are therefore mutually equiangular. § 850

Hence we can show that R and S, the polar triangles of R' and S', are mutually equilateral.

Therefore R and S are equal or symmetrical, according to the arrangement of their homologous parts.

§§ 873 III, 874 III

Q. E. D.

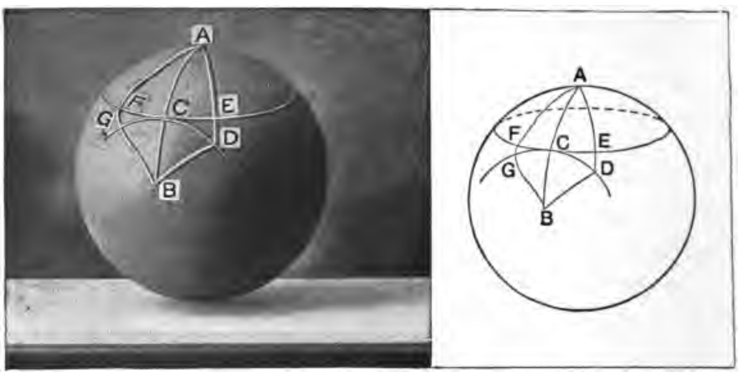
884. COR. *If two triedral angles have their diedral angles equal each to each:*

I. *They are equal, when the equal diedral angles are arranged in the same order in both triedral angles.*

II. *They are symmetrical, when the equal diedral angles are arranged in opposite order in the two triedral angles.*

PROPOSITION XXXII. THEOREM

885. *The shortest line that can be drawn on the surface of a sphere between two points is the arc of a great circle, not greater than a semi-circumference, joining those points.*



GIVEN—an arc of a great circle AB, not greater than a semi-circumference, joining the points A and B on a spherical surface.

TO PROVE—that AB is the shortest line that can be drawn on the surface between A and B.

CASE I. *When AB is less than a semi-circumference.*

Let C be any point of AB.

With A and B as poles describe circumferences whose polar distances are AC and BC.

These circumferences have only the point C in common.

For, let D be any other point of the circumference whose pole is B.

Draw the great-circle-arcs AD and BD and let AD meet the circumference whose pole is A in E.

Then $AD+BD>AC+BC$. § 864

But $BD=BC$ and $AC=AE$. § 817

Hence $AD>AE$.

Therefore D lies outside the small circle whose pole is A, and the two small circles have only the point C in common.

Now we will prove that the shortest line on the surface between A and B must pass through C.

Let $AFGB$ be any line on the surface between A and B that does not pass through C.

It must cut the small circles in separate points F and G.

Now, whatever may be the nature of the line AF, an equal line can be drawn on the surface between A and C.

[This can be shown by supposing the spherical surface to revolve on the axis of the small circle FCE, so that F will move along the small circle to C, while A remains fixed.]

Similarly a line equal to BG can be drawn from B to C.

There will then lie between A and B and passing through C a line less than $AFGB$ by the portion FG.

We have now proved that through C can be drawn a line joining A and B less than any line joining A and B that does not pass through C.

Hence the shortest line must pass through C.

But C is *any* point in the arc AB.

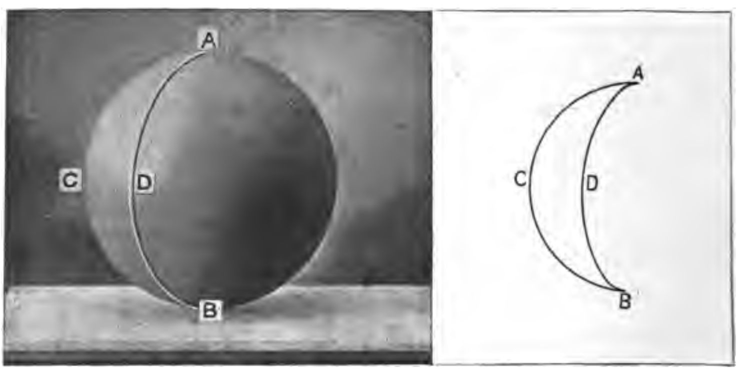
Therefore the shortest line between A and B must pass through every point of the arc AB and hence must coincide with that arc. Q. E. D.

CASE II. *When AB is a semi-circumference.*

We can show as above that any portion of the shortest line joining A and B must be an arc of a great circle, and that therefore the whole must be an arc of a great circle. Q. E. D.

MEASUREMENT OF SPHERICAL FIGURES

886. *Defs.*—A **lune** is a portion of a spherical surface bounded by two semi-circumferences of great circles; as $ACBDA$.

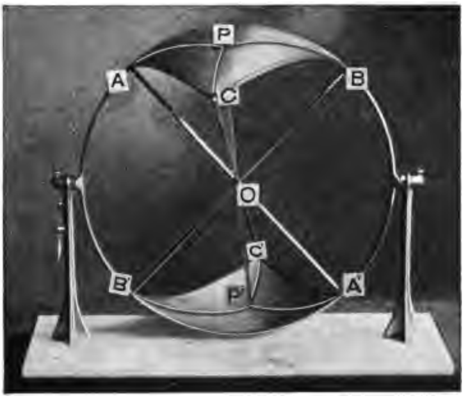


The **angle of a lune** is the angle formed by its bounding arcs.

Thus CAD is the angle of the lune $ACBDA$.

PROPOSITION XXXIII. THEOREM

887. *Two symmetrical spherical triangles are equivalent.*



GIVEN two symmetrical triangles ABC and $A'B'C'$.

TO PROVE area $ABC =$ area $A'B'C'$.

Let P be the pole of the small circle passing through A, B, and C, and draw the great-circle-arcs PA, PB, and PC.

Then $PA = PB = PC$. § 817

Now place the two triangles vertically opposite to each other and draw the diameter POP'. § 853

Also draw the great-circle-arcs $P'A'$, $P'B'$, and $P'C'$.

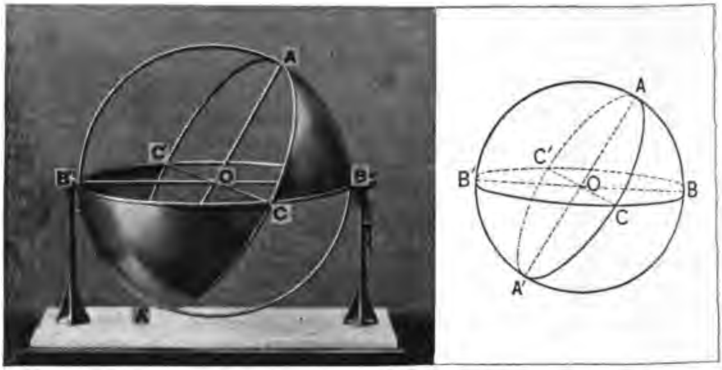
The vertical triangles PBC and $P'B'C'$ are symmetrical and isosceles and therefore equal. § 855

Similarly $PCA = P'C'A'$ and $PAB = P'A'B'$.

That is, the three parts of ABC are respectively equal to three parts of $A'B'C'$.

Therefore area $ABC =$ area $A'B'C'$. Q. E. D.

888. COR. I. *If two semi-circumferences of great circles* BCB' and ACA' *intersect on the surface of a hemisphere, the sum of the areas of the two opposite spherical triangles* ABC *and* $CA'B'$ *is equal to the area of a lune whose angle is equal to* BCA.

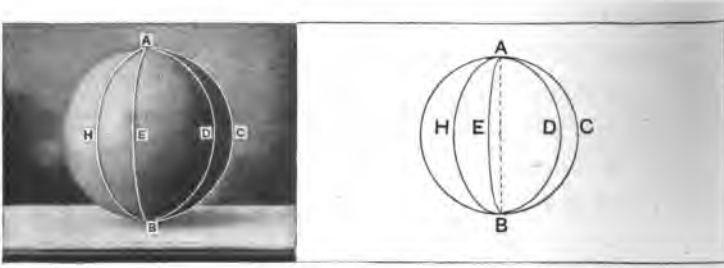


Hint.—Area ABC + area $CA'B'$ = area $A'B'C'$ + area $CA'B'$.

889. COR. II. *Two symmetrical spherical polygons are equivalent.*

PROPOSITION XXXIV. THEOREM

890. *Two lunes on the same sphere are equal, if their angles are equal.*



GIVEN—two lunes *ADBC* and *AEBH* on the same sphere, their angles *DAC* and *HAE* being equal.

TO PROVE that the lunes are equal.

Since the angles DAC and HAE are equal, the plane angles of the diedral angles $DABC$ and $HABE$ are equal. § 836 I

Hence these diedral angles are equal. § 572

They can therefore be superposed.

At the same time the lunes coincide.

Therefore the lunes are equal. Q. E. D.

PROPOSITION XXXV. THEOREM

891. *Two lunes on the same sphere are to each other as their angles.*

GIVEN—the lunes $ADBE$ and $ACBD$, whose angles are DAE and CAD.

TO PROVE $\frac{ADBE}{ACBD}=\frac{DAE}{CAD}$.

CASE I. *When the angles are commensurable* (Fig. 1).

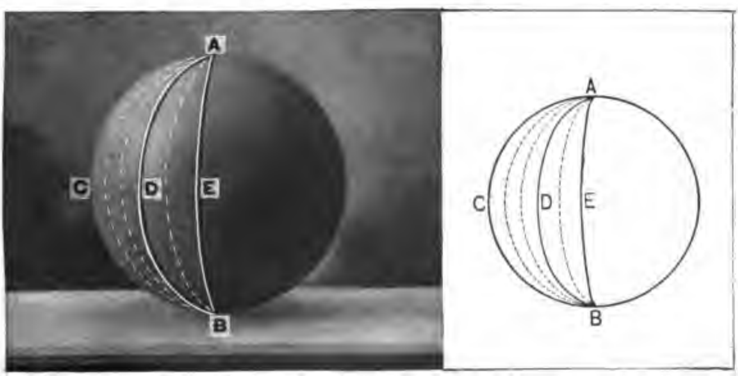


FIG. 1

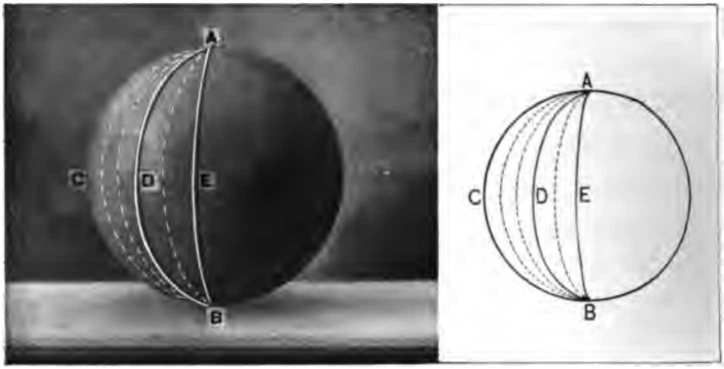


FIG. 1

Suppose a common measure of DAE and CAD to be contained twice in DAE and 3 times in CAD.

Then $$\frac{DAE}{CAD}=\frac{2}{3}.$$ § 180

Draw from A to B semi-circumferences of great circles dividing the angles DAE and CAD into parts each equal to their common measure.

The little lunes thus formed are all equal. § 890

Of these lunes $ADBE$ contains 2 and $ACBD$ 3.

Hence $$\frac{ADBE}{ACBD}=\frac{2}{3}.$$ § 180

Therefore $$\frac{ADBE}{ACBD}=\frac{DAE}{CAD}.$$ Q. E. D.

CASE II. *When the angles are incommensurable* (Fig 2).

Divide CAD into any number of equal parts by arcs of great circles drawn from A to B.

Apply one of these parts to DAE as many times as it will be contained in it, the final bounding arc taking the position $AE'B$.

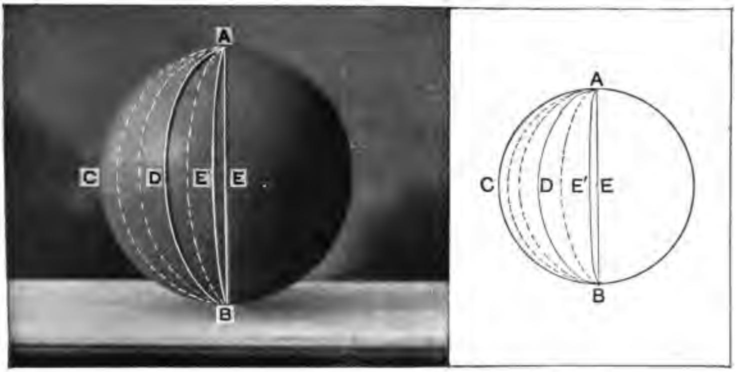


FIG. 2

Since the angles are incommensurable there will be a remainder $E'AE$ less than one of these parts.

Now the angles DAE' and CAD are commensurable.

Therefore $$\frac{ADBE'}{ACBD}=\frac{DAE'}{CAD}.$$ Case I

Let the number of parts into which CAD is divided be indefinitely increased.

Then the angle DAE' will approach DAE as a limit. § 185

The lune $ADBE'$ will approach $ADBE$ as a limit.

Also $\frac{DAE'}{CAD}$ will approach $\frac{DAE}{CAD}$ as a limit. § 190

And $\frac{ADBE'}{ACBD}$ will approach $\frac{ADBE}{ACBD}$ as a limit.

Therefore $$\frac{ADBE}{ACBD}=\frac{DAE}{CAD}.$$ § 186

Q. E. D.

892. COR. I. *A lune is to the surface of the sphere on which it lies as the angle of the lune is to four right angles.*

Hint.—The surface of a sphere may be regarded as the limit of a lune whose angle approaches four right angles as a limit.

893. COR. II. Let A denote the angle of a lune measured in the right angle as a unit and L its surface measured in the tri-rectangular triangle as a unit.

Then the area of the spherical surface will be 8. § 882

Hence $$\frac{L}{8}=\frac{A}{4}.$$ § 892

Therefore $$L=2A.$$

That is, *if the unit angle is the right angle and the unit surface the tri-rectangular triangle, a lune is measured by twice its angle.*

894. *Def.*—The **spherical excess** of a spherical triangle is the excess of the sum of its angles over two right angles.

Denoting the angles by A, B, C, and the spherical excess by E, we have, taking the right angle as the unit angle,

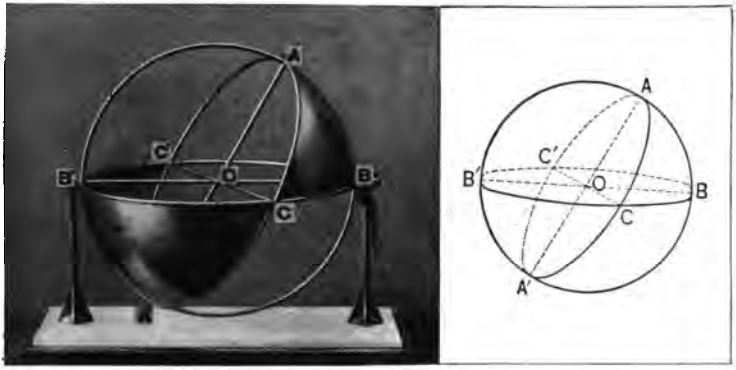
$$E=A+B+C-2.$$

Thus, if the angles of a spherical triangle are 45°, 60°, 135°, its spherical excess is

$$\left(\frac{45}{90}+\frac{60}{90}+\frac{135}{90}-2\right) \text{ right angles}=\frac{2}{3} \text{ right angle.}$$

PROPOSITION XXXVI. THEOREM

895. *If the unit angle is the right angle and the unit surface the tri-rectangular triangle, the area of a spherical triangle is measured by its spherical excess.*



GIVEN the spherical triangle ABC.

TO PROVE area $ABC = A + B + C - 2$,

the unit angle being the right angle and the unit surface the surface of the tri-rectangular triangle.

Complete the circumference of which AB is an arc, and let BC and AC intersect it again in B' and A'.

Then, since BCA and $B'CA$ together form a lune whose angle is B,

area BCA + area $B'CA = 2B$. § 893

Similarly, area CAB + area $A'CB = 2A$.

Also the triangles ABC and $CA'B'$ are together equal to a lune whose angle is C. § 888

Hence area ABC + area $CA'B' = 2C$.

Now the sum of the areas of ABC, $B'CA$, $A'CB$, and $CA'B'$ is the area of the surface of a hemisphere, which with the adopted unit is 4.

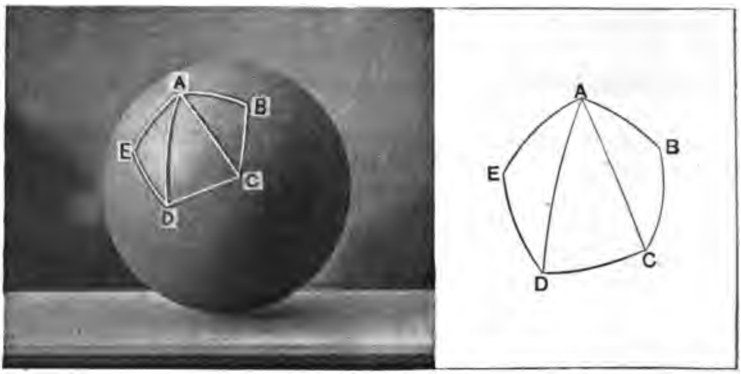
Hence, adding the three equations above, we have,

$$2 \text{ area } ABC + 4 = 2A + 2B + 2C.$$

Therefore area $ABC \quad = A + B + C - 2$. Q. E. D.

PROPOSITION XXXVII. THEOREM

896. *If the unit angle is the right angle, and the unit surface the tri-rectangular triangle, the area of a spherical polygon is measured by the sum of its angles minus twice the number of its sides less two.*



GIVEN—the spherical polygon $ABCDE$. Denote its area measured in tri-rectangular triangles by K; the sum of its angles measured in right angles by S; and the number of its sides by n.

TO PROVE $\quad K = S - 2(n-2).$

Divide the polygon into triangles by diagonals drawn from any vertex A.

The area of each triangle is measured by the sum of its angles less two. § 895

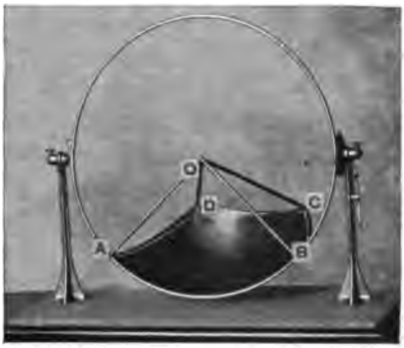
The number of triangles is $n-2$, there being one for every side except the sides intersecting in A.

Hence the area of the polygon is measured by the sum of the angles of all the triangles minus $2(n-2)$.

But the sum of the angles of all the triangles is equal to the sum of the angles of the polygon.

Therefore $K = S - 2(n-2)$. Q. E. D.

897. *Defs.*—A **spherical pyramid** is a solid bounded by a spherical polygon and the planes of its sides; as $O-ABCD$.



The centre of the sphere is called the **vertex** of the spherical pyramid, and the spherical polygon its **base.**

898. *Defs.*—A **spherical ungula,** or **wedge,** is a solid bounded by a lune and the planes of its bounding arcs.

The lune is called the **base** of the ungula; the diameter in which the bounding planes meet is its **edge.**

The angle of the bounding lune is also called the **angle of the ungula.**

899. The proofs of the following theorems relating to spherical pyramids and ungulas correspond so closely to the proofs of the corresponding theorems relating to spherical polygons and lunes that they are left as exercises for the student.

I. *Two symmetrical triangular spherical pyramids are equivalent.*

II. *If two great semicircles $BCB'O$ and $ACA'O$* (see Fig. § 888) *intersect in a hemisphere, the sum of the volumes of the two opposite spherical pyramids $O - ABC$ and $O - CA'B'$ is equal to the volume of an ungula whose angle is equal to BCA.*

III. *Two symmetrical spherical pyramids are equivalent.*

IV. *Two ungulas in the same sphere are equal if their angles are equal.*

V. *Two ungulas in the same sphere are to each other as their angles.*

VI. *An ungula is to the sphere of which it is a part as its angle is to four right angles.*

If the unit angle is the right angle and the unit solid the tri-rectangular spherical pyramid (that whose base is the tri-rectangular spherical triangle):

VII. *An ungula is measured by twice its angle.*

VIII. *The volume of a triangular spherical pyramid is measured by the spherical excess of its base.*

IX. *The volume of a spherical pyramid is measured by the sum of the angles of its base minus twice the number of its sides less two.*

The following theorems are simple corollaries of the preceding:

X. *Two triangular spherical pyramids in the same sphere are to each other as their bases.*

XI. *Any two spherical pyramids in the same sphere are to each other as their bases.*

XII. *Any spherical pyramid is to the sphere of which it is a part as its base is to the surface of the sphere.*

PROBLEMS OF DEMONSTRATION

900. *Exercise.*—The intersection of two spherical surfaces is the circumference of a circle whose plane is perpen-

dicular to the straight line joining the centres of the two spherical surfaces, and whose centre is in that line.

901. *Exercise.*—If from a point without a sphere a tangent and a secant line be drawn, the square of the tangent is equal to the product of the whole secant and its external segment.

902. *Exercise.*—If the centres of three spheres do not lie in the same straight line, their surfaces cannot have more than two points in common. These points lie in a straight line perpendicular to the plane of centres and at equal distances from this plane on opposite sides.

903. *Exercise.*—From a given point on the surface of a sphere, and not on a given great circle, but two great-circle-arcs can be drawn perpendicular to the given great circle; and these are the shortest and longest great-circle-arcs that can be drawn from the point to the given great circle.

904. *Exercise.*—If any number of lines in space meet in a point, the feet of the perpendiculars drawn to these lines from another point lie on the surface of a sphere.

905. *Exercise.*—If from a point within a spherical triangle arcs of great circles are drawn to the extremities of one side, the sum of these arcs is less than the sum of the two other sides of the triangle.

906. *Exercise.*—Any point in the bisector of a spherical angle is equally distant from the sides of the angle.

907. *Exercise.*—The bisectors of the angles of a spherical triangle meet in a point which is equally distant from the sides of the triangle.

908. *Exercise.*—The three medians of a spherical triangle meet in a point.

909. *Exercise.*—The perpendicular bisectors of the sides of a spherical triangle meet in a point.

910. *Exercise.*—If a, b, c are the sides of a spherical triangle and a', b', c' the corresponding sides of the polar triangle, if $a>b>c$, then $a'<b'<c'$.

911. *Exercise.*—Spherical triangles on equal spheres have equal areas if their polar triangles have equal perimeters.

LOCI

912. *Exercise.*—Find the locus of a point at a given distance from an indefinite straight line.

913. *Exercise.*—Find the locus of a point at a given distance from a straight line of definite length.

914. *Exercise.*—Find the locus of a point whose distance from a fixed straight line is in a given ratio to its distance from a fixed plane perpendicular to that line.

915. *Exercise.*—Find the locus of a point from which tangent lines drawn to three mutually intersecting spheres are equal.

916. *Exercise.*—Find the locus of the centre of a sphere which is tangent to three given planes.

917. *Exercise.*—Find the locus of a point in space the ratio of whose distances from two given points is constant.

918. *Exercise.*—Find the locus of the centre of the section of a given sphere made by a plane passing through a given point.

919. *Exercise.*—From a fixed point straight lines are drawn to the surface of a sphere. Find the locus of the points which divide these lines in a given ratio.

920. *Exercise.*—Find the locus of a point on the surface of a sphere equidistant from two given points on the surface.

921. *Exercise.*—Find the locus of a point on the surface of a sphere equidistant from three given points on the surface.

922. *Exercise.*—If the angles adjacent to a side of a spherical triangle are supplementary, the locus of the opposite vertex is a great circle.

PROBLEMS OF CONSTRUCTION

923. *Exercise.*—Through a given straight line not intersecting a sphere pass a plane tangent to the sphere.

924. *Exercise.*—Construct a spherical surface of given radius:

(*a.*) Passing through three given points.

(*b.*) Passing through two given points and tangent to a given plane.

(*c.*) Passing through two given points and tangent to a given sphere.

(*d.*) Passing through a given point and tangent to two given planes.

(*e.*) Passing through a given point and tangent to two given spheres.

(*f.*) Tangent to three given spheres.

(*g.*) Tangent to a given plane and two given spheres.

925. *Exercise.*—Bisect a given arc of a great circle.

926. *Exercise.*—Through a given point on a sphere draw a great circle tangent to a given small circle.

PROBLEMS FOR COMPUTATION

927. (1.) The radius of a sphere is 25 in. Find the area of a section made by a plane 10 in. distant from its centre.

(2.) What is the radius of a sphere inscribed in a regular tetraedron whose total area is 4 sq. m.?

(3.) What is the radius of a spherical surface passing through four points each of which is 9 cm. distant from the other three?

(4.) If the volume of a sphere is 12 cu. m., what is the volume of a spherical wedge the angles of whose base are 40°?

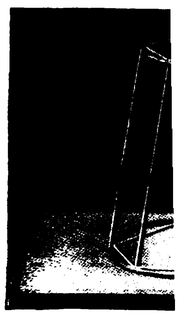
(5.) If the area of a spherical surface is 100 sq. ft., what is the area of a spherical triangle whose angles are 30°, 120°, and 150°?

(6.) The volume of a sphere is 1000 cu. in. What is the volume of a spherical pyramid the angles of whose base are 30°, 90°, 130°, and 160°?

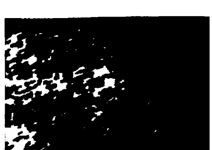
(

MEASUREM

928. *Def*
lateral edges
in the planes



929. *Def.*-
when its later
bases are in th



930. *Def.*—A **right section** of a cylinder is a section made by a plane perpendicular to its elements.

931. *Remark.*—From the preceding definitions it follows immediately that the bases of an inscribed prism are inscribed in the bases of the cylinder; the bases of a circumscribed prism are circumscribed about the bases of the cylinder; and that a plane forming a right section of a cylinder forms a right section of every inscribed and every circumscribed prism.

932. *Def.*—The **lateral area** of a cylinder is the area of its lateral surface.

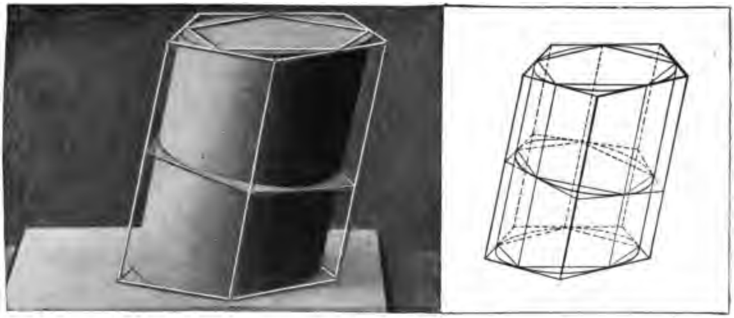
PROPOSITION I. THEOREM

933. *If the number of lateral faces of a prism inscribed in or circumscribed about a cylinder be indefinitely increased so that each one becomes indefinitely small, then*

I. *Any right section of the prism approaches a right section of the cylinder as a limit.*

II. *The lateral area of the prism approaches the lateral area of the cylinder as a limit.*

III. *The volume of the prism approaches the volume of the cylinder as a limit.*



Proof.—I. A plane which forms a right section of the prism will also form a right section of the cylinder. § 931

When the number of lateral faces of the prism is indefinitely increased so that each one becomes indefinitely small, the number of sides of the right section will be indefinitely increased, and each will become indefinitely small.

Therefore the right section of the prism approaches the right section of the cylinder as a limit. § 490

Q. E. D.

II. The lateral surface of the prism can be generated by a straight line moving about its right section as a directrix, provided this line remains parallel to the lateral edges and is terminated by the two bases. § 632

As the number of lateral faces increases indefinitely, the directrix of this line approaches the right section of the cylinder as a limit.

Hence the limit of the surface generated by this line is the surface generated by it when the directrix is the perimeter of the right section of the cylinder.

But this surface is the lateral surface of the cylinder. § 766

Therefore the limit of the lateral area of the prism is the lateral area of the cylinder. Q. E. D.

III. Let B', B'' be the respective bases of a circumscribed and corresponding inscribed prism, V', V'' their respective volumes, and H their common altitude.

Then $\qquad V' = B' \times H$, and $V'' = B'' \times H$. $\qquad$ § 676

Hence $\qquad V' - V'' = (B' - B'') \times H$.

Now by increasing indefinitely the number of lateral faces of the prisms, and consequently the number of sides of their bases, the difference $B' - B''$ can be made as small as we please. § 490

Hence $(B' - B'') \times H$ can be made as small as we please. § 187

Hence its equal $V' - V''$ can be made as small as we please.

But the volume of the cylinder is always intermediate between V' and V''. Ax. 10

Therefore the difference between the volume of the cylinder and either V' or V'' can be made as small as we please.

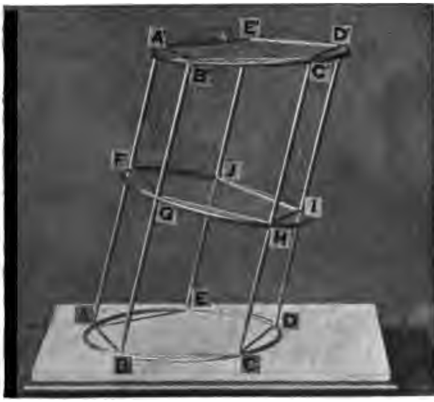
But V' and V'' can never equal the volume of the cylinder. Ax. 10

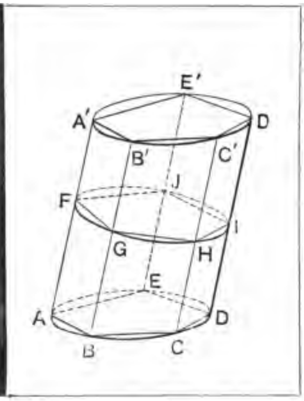
Therefore the volume of the cylinder is the common limit of V' and V''. § 185

Q. E. D.

PROPOSITION II. THEOREM

934. *The lateral area of a cylinder is equal to the product of the perimeter of a right section and an element.*





GIVEN—the cylinder AD', of which P is the perimeter of the right section $FGHIJ$, E an element, and S the lateral area.

TO PROVE $S = P \times E$.

Inscribe in the cylinder a prism. Let P' be the perimeter of its right section and S' its lateral area.

Its lateral edge is equal to E. § 545

Hence $S' = P' \times E$. § 649

Now let the number of lateral faces of the prism be indefinitely increased.

Then S' approaches S as a limit, § 933 II

P' approaches P as a limit, § 933 I

and $P' \times E$ approaches $P \times E$ as a limit. § 189

Therefore $S = P \times E$. § 186

Q. E. D.

935. *Def.*—The **altitude** of a cylinder is the perpendicular distance between its bases.

936. COR. I. *The lateral area of a right cylinder is equal to the product of the perimeter of its base by its altitude.*

937. COR. II. Let H denote the altitude, R the radius, S the lateral area, and T the total area of a cylinder of revolution.

Then $\quad S=2\pi RH,$

and $\quad T=2\pi RH+2\pi R^2=2\pi R(H+R).$

938. *Def.*—**Similar cylinders of revolution** are cylinders formed by the revolution of similar rectangles about homologous sides.

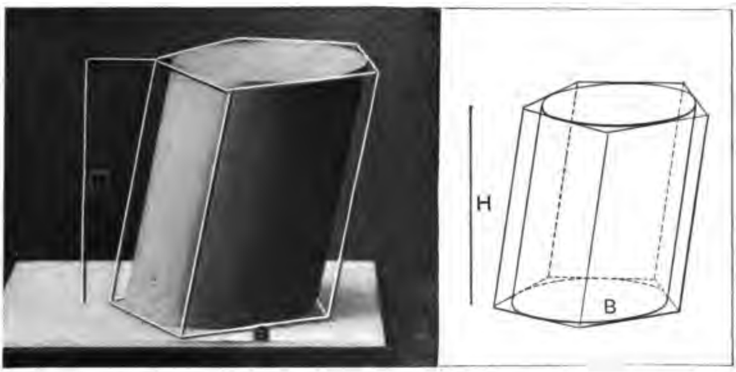
939. COR. III. *The lateral areas, or the total areas, of two similar cylinders of revolution are to each other as the squares of their altitudes, or as the squares of their radii.*

OUTLINE PROOF: $\frac{S}{s}=\frac{2\pi RH}{2\pi rh}=\frac{R}{r}\times\frac{H}{h}=\frac{H}{h}\times\frac{H}{h}=\frac{H^2}{h^2}=\frac{R^2}{r^2}$.

$$\frac{T}{t}=\frac{2\pi R(H+R)}{2\pi r(h+r)}=\frac{R}{r}\times\frac{H+R}{h+r}=\frac{H}{h}\times\frac{H}{h}=\frac{H^2}{h^2}=\frac{R^2}{r^2}.$$

PROPOSITION III. THEOREM

940. *The volume of a cylinder is equal to the product of its base and altitude.*



GIVEN—a cylinder, of which B is the base, H the altitude, and V the volume.

TO PROVE $V=B\times H$.

Circumscribe about the cylinder a prism. Denote its base by B' and its volume by V'.

Its altitude is H. § 545

Hence $V'=B'\times H$. § 676

Now let the number of lateral faces of the prism be indefinitely increased.

Then V' approaches V as a limit, § 933 III

B' approaches B as a limit, § 490

and $B' \times H$ approaches $B \times H$ as a limit. § 189

Therefore $V = B \times H.$ § 186

Q. E. D.

941. COR. I. Let H be the altitude, R the radius, and V the volume of a circular cylinder.

Then $V = \pi R^2 H.$

942. COR. II. *The volumes of two similar cylinders of revolution are to each other as the cubes of their altitudes, or as the cubes of their radii.*

OUTLINE PROOF: $\dfrac{V}{v} = \dfrac{\pi R^2 H}{\pi r^2 h} = \dfrac{R^2}{r^2} \times \dfrac{H}{h} = \dfrac{H^2}{h^2} \times \dfrac{H}{h} = \dfrac{H^3}{h^3} = \dfrac{R^3}{r^3}.$

THE CONE

943. *Def.*—A pyramid is **inscribed in a cone** when its lateral edges are elements of the cone and its base is in the plane of the base of the cone.

944. *Def.*—A pyramid is **circumscribed about a cone** when its lateral faces are tangent to the cone and its base is in the plane of the base of the cone.

945. *Remark.*—From these definitions it follows immediately that the base of an inscribed pyramid is inscribed in the base of the cone and that the base of a circumscribed pyramid is circumscribed about the base of the cone.

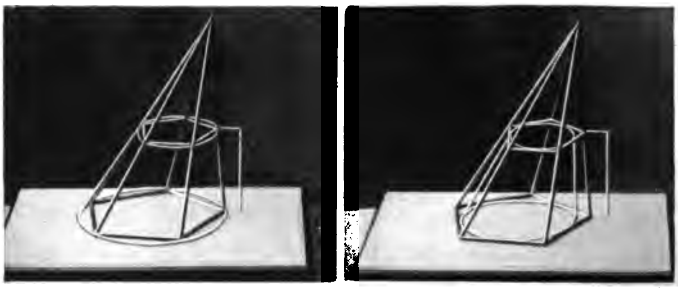
946. *Defs.*—A **truncated cone** is the portion of a cone contained between its base and a plane cutting all its elements.

The base of the cone and the section made by the cutting plane are called the **bases** of the truncated cone.

FRUSTUM TRUNCATED CONE

947. *Def.*—A **frustum of a cone** is a truncated cone whose bases are parallel.

948. *Def.*—If a pyramid is inscribed in or circumscribed about a cone, a plane which cuts from the cone a truncated cone cuts from the pyramid a frustum of a pyramid which may be said to be **inscribed** in or **circumscribed** about the frustum of a truncated cone.



949. *Def.*—The **lateral area** of a cone is the area of its lateral surface.

PROPOSITION IV. THEOREM

950. *If the number of lateral faces of a pyramid inscribed in or circumscribed about a cone be indefinitely increased so that each one becomes indefinitely small, then*

I. *Any section of the pyramid approaches the section of the cone by the same plane as a limit.*

II. *The lateral area of the pyramid approaches the lateral area of the cone as a limit.*

III. *The volume of the pyramid approaches the volume of the cone as a limit.*

The proof of this proposition is analogous to that of Proposition I., and is therefore left to the student.

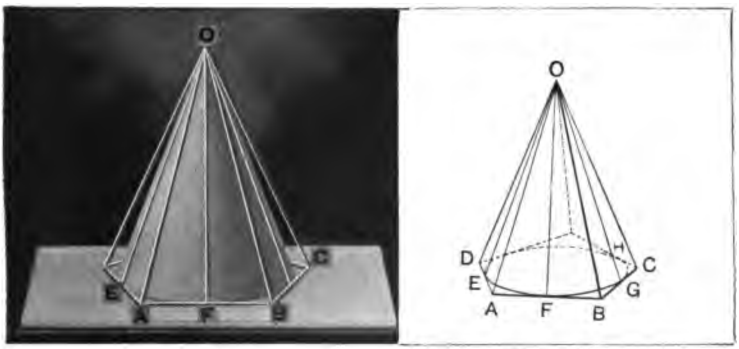
951. *Remark.*—The proposition obtained from the preceding by substituting the words "frustum of a pyramid" and "frustum of a cone" for "pyramid" and "cone" can be proved in the same way.

952. *Def.* — Any element of a cone of revolution is called its **slant height.**

953. *Exercise.*—Prove that the slant height of a regular pyramid circumscribed about a cone of revolution is equal to the slant height of the cone of revolution.

PROPOSITION V. THEOREM

954. *The lateral area of a cone of revolution is equal to one-half the product of the circumference of its base by its slant height.*



GIVEN—the cone of revolution O-$EFGH$. Denote its slant height OE by L, the circumference of its base by C, and its lateral area by S.

TO PROVE $S=\frac{1}{2}C\times L$.

Circumscribe about the cone a regular pyramid. Denote the perimeter of its base by C' and its lateral area by S'.

Its slant height will also be L. § 953

Hence $S'=\frac{1}{2}C'\times L$. § 688

Now let the number of lateral faces of the regular pyramid be indefinitely increased.

Then S' approaches S as a limit. § 950 II

And C' approaches C as a limit. § 490

Hence $\frac{1}{2}C' \times L$ approaches $\frac{1}{2}C \times L$ as a limit. § 189

Therefore $S = \frac{1}{2}C \times L$. § 186

Q. E. D.

955. COR. I. Let R denote the radius, L the slant height, S the lateral area, and T the total area of a cone of revolution.

Then $S = \frac{1}{2}2\pi R \times L = \pi RL$.

And $T = \pi RL + \pi R^2 = \pi R(L+R)$.

956. COR. II. The formula for the lateral area may be written $S = 2\pi \frac{R}{2} \times L$.

Now, if K is the radius of a section half-way between the vertex and base, $K = \frac{1}{2}R$.

Therefore $S = 2\pi K \times L$.

That is, *the lateral area of a cone of revolution is equal to the circumference of a section half-way between its vertex and base multiplied by its slant height.*

957. *Def.*—The **altitude** of a cone is the perpendicular distance from its vertex to its base.

958. *Def.* — **Similar cones of revolution** are cones formed by the revolution of similar right triangles about homologous sides.

959. COR. III. *The lateral areas, or the total areas, of two similar cones of revolution are to each other as the squares of their slant heights, or as the squares of their altitudes, or as the squares of the radii of their bases.*

Hint.—The method of proof is the same as that followed in § 939.

960. *Def.*—The portion of an element of a cone of revolution included between the bases of a frustum is called **the slant height** of the frustum.

961. *Exercise.*—Prove that the slant height of a frustum of a regular pyramid which is circumscribed about a frustum of a cone of revolution is equal to the slant height of the frustum of a cone.

PROPOSITION VI. THEOREM

962. *The lateral area of a frustum of a cone of revolution is equal to half the sum of the circumferences of its bases multiplied by its slant height.*

GIVEN a frustum of a cone of revolution.

Denote the circumferences of its bases by C and c, its slant height by L, and its lateral area by S.

TO PROVE $S=\frac{1}{2}(C+c)\times L$.

Circumscribe about the frustum a frustum of a regular pyramid.

Denote the perimeters of its bases by C' and c', and its lateral area by S'. Its slant height will also be L. § 961

Hence $S'=\frac{1}{2}(C'+c')\times L$. § 693

Now let the number of lateral faces of the frustum of a regular pyramid be indefinitely increased.

Then S' approaches S as a limit, § 951

$C'+c'$ approaches $C+c$ as a limit, § 490

and $\frac{1}{2}(C'+c')\times L$ approaches $\frac{1}{2}(C+c)\times L$ as a limit. § 189

Therefore $S=\frac{1}{2}(C+c)\times L$. § 186

Q. E. D.

963. COR. I. If R and r are the radii of the bases, L the slant height, and S the lateral area of a frustum of a cone of revolution,

$$S=\tfrac{1}{2}(2\pi R+2\pi r)\times L=\pi(R+r)\times L.$$

964. COR. II. The last formula may be written

$$S=2\pi\frac{R+r}{2}\times L.$$

If K is the radius of a section half-way between the bases of the frustum,

$$K=\frac{R+r}{2}.$$

Hence $$S=2\pi K\times L.$$

That is, *the lateral area of a frustum of a cone of revolution is equal to the circumference of a section half-way between its bases multiplied by its slant height.*

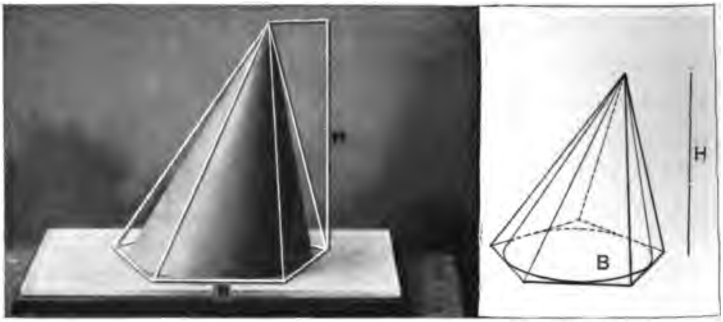
PROPOSITION VII. THEOREM

965. *The volume of a cone is equal to one-third the product of its base and altitude.*

GIVEN—any cone, of which B is the base, H the altitude, and V the volume.

TO PROVE $$V=\tfrac{1}{3}B\times H.$$

Circumscribe about the cone a pyramid. Denote its base by B', and its volume by V'. Its altitude is H.



Then $V' = \frac{1}{3}B' \times H.$ § 704

Now let the number of lateral faces of the pyramid be indefinitely increased.

Then V' approaches V as a limit, § 950 III

B' approaches B as a limit, § 490

and $\frac{1}{3}B' \times H$ approaches $\frac{1}{3}B' \times H$ as a limit. § 189

Therefore $V = \frac{1}{3}B \times H.$ Q. E. D.

966. COR. I. If the base of the cone is a circle of radius R,

$$V = \frac{1}{3}\pi R^2 H.$$

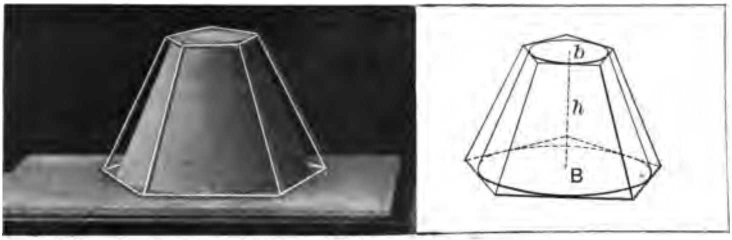
967. COR. II. *The volumes of two similar cones of revolution are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.*

Hint.—The method of proof is the same as that followed in § 942.

968. *Def.*—The **altitude** of a frustum of a cone is the perpendicular distance between its bases.

PROPOSITION VIII. THEOREM

969. *A frustum of a cone is equivalent to the sum of three cones whose common altitude is the altitude of the frustum and whose bases are the lower base, the upper base, and a mean proportional between the bases of the frustum.*



GIVEN—a frustum of a cone. Denote its bases by B and b, its altitude by h, and its volume by V.

TO PROVE—$V=\frac{1}{3}h(B+b+\sqrt{B\times b})$, which is the algebraic statement of the theorem.

Circumscribe about the frustum of a cone a frustum of a pyramid. Denote its bases by B' and b', and its volume by V'.

Its altitude will be h. § 565

Hence $V'=\frac{1}{3}h(B'+b'+\sqrt{B'\times b'})$. § 713

Now let the number of lateral faces of the frustum of a pyramid be indefinitely increased.

Then V' approaches V as a limit, § 951

B' approaches B as a limit, § 490

b' approaches b as a limit,

and $\frac{1}{3}h(B'+b'+\sqrt{B'\times b'})$ approaches $\frac{1}{3}h(B+b+\sqrt{B\times b})$.

Therefore $V=\frac{1}{3}h(B+b+\sqrt{B\times b})$. § 186

Q. E. D.

970. COR. If the frustum is the frustum of a circular cone, let R and r be the radii of its bases.

Then $B=\pi R^2$, $b=\pi r^2$, $\sqrt{B\times b}=\pi Rr$.

Therefore $V=\frac{1}{3}\pi h(R^2+r^2+Rr)$.

THE SPHERE

971. *Defs.*—A **zone** is a portion of the surface of a sphere bounded by the circumferences of two circles whose planes are parallel.

The bounding circumferences are called the **bases**, and the perpendicular distance between their planes the **altitude** of the zone.

972. *Def.*—If the plane of one bounding circumference is tangent to the sphere, the zone is called a **zone of one base.**

973. *Defs.*—A **spherical segment** is a portion of a sphere contained between two parallel planes.

The bounding circles are called the **bases**, and the perpendicular distance between their planes the **altitude** of the segment.

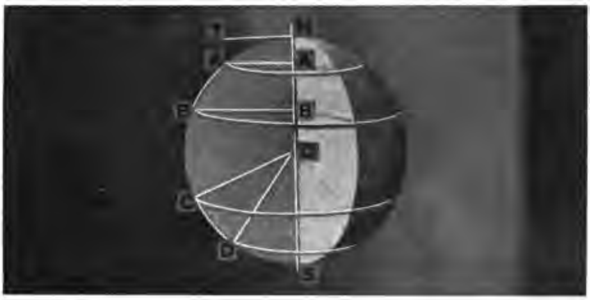
974. *Def.*—A **spherical segment of one base** is a spherical segment one of whose bounding planes is tangent to the sphere.

The curved surface of a spherical segment is a zone.

975. *Defs.*—If a semicircle is revolved about its diameter as an axis, the solid generated by any sector of the semicircle is called a **spherical sector.**

The zone generated by the base of the sector of the semicircle is called the **base** of the spherical sector.

976. *Remarks.*—Suppose a sphere generated by the revolution of the semicircle HAS about its diameter HS as an axis. Let AA' and BB' be two lines perpendicular to HS, and let OC and OD be radii of the semicircle.



Then the arc AB generates a zone whose altitude is $A'B'$; the points A and B generate the bases of the zone.

The arc HA generates a zone of one base.

The figure $AA'B'B$ generates a spherical segment whose altitude is $A'B'$; the lines AA' and BB' generate the bases of the spherical segment.

The figure HAA' generates a spherical segment of one base.

The sector COD of the semicircle generates a spherical sector. This spherical sector is bounded by three curved surfaces, namely: the two conical surfaces generated by the radii OC and OD, and the zone generated by the arc CD.

PROPOSITION IX. LEMMA

977. *The area of the surface generated by a straight line revolving about an axis in its plane (not crossing the straight line) is equal to the projection of the line on the axis multiplied by the circumference of the circle whose radius is the perpendicular to the line drawn at its middle point and terminated in the axis.*

FIG. 1

FIG. 2

FIG. 3

GIVEN—the straight lines AB and XY in the same plane, XY not crossing AB. Let S denote the area of the surface generated by revolving AB about XY as an axis.

Draw a perpendicular MO to AB at its middle point M cutting XY in O, and let $A'B'$ be the projection of AB on XY.

TO PROVE $S = A'B' \times 2\pi MO$.

CASE I. *When AB is parallel to XY* (Fig. 1).

The surface generated in this case is the lateral surface of a cylinder of revolution. § 778

Hence $S = AB \times 2\pi BB'$. § 937

Or $S = A'B' \times 2\pi MO$. § 117

Q. E. D.

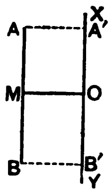


FIG. 1

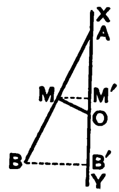


FIG. 2

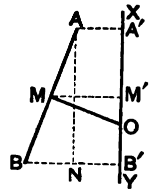


FIG. 3

CASE II. *When one end A of AB is in XY* (Fig. 2).

The surface generated in this case is the lateral surface of a cone of revolution. § 794

Draw MM' perpendicular to XY.

Then $S = AB \times 2\pi MM'$. § 956

The triangles $AB'B$ and $MM'O$ are similar. § 286

Hence $\frac{AB'}{AB} = \frac{MM'}{MO} = \frac{2\pi MM'}{2\pi MO}$. § 274

Hence $AB \times 2\pi MM' = AB' \times 2\pi MO$. § 250

Therefore $S = AB' \times 2\pi MO$. Q. E. D.

CASE III. *When AB is not parallel to XY and does not meet XY* (Fig. 3).

The surface generated in this case will be that of a frustum of a cone of revolution.

Draw MM' perpendicular to XY and AN perpendicular to BB'.

Then $S = AB \times 2\pi MM'$. § 964

The triangles ANB and $MM'O$ are similar. § 286

Hence $\frac{AN}{AB} = \frac{MM'}{MO} = \frac{2\pi MM'}{2\pi MO}$.

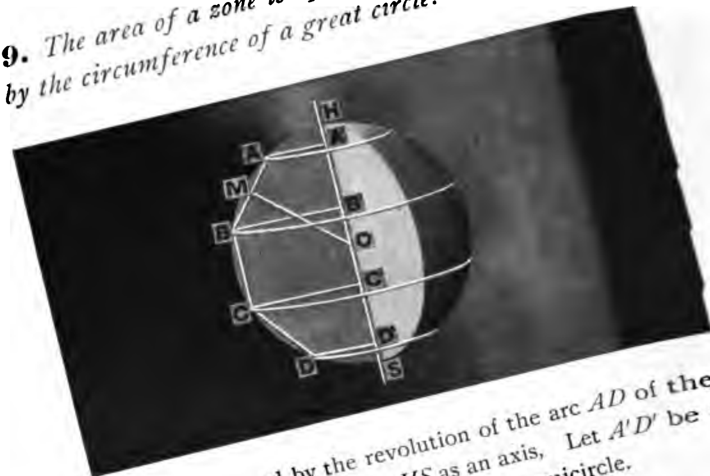
Hence $AB \times 2\pi MM' = AN \times 2\pi MO = A'B' \times 2\pi MO$.

Therefore $S = A'B' \times 2\pi MO$. Q. E. D.

978. *Def.*—A **broken line** is a line which is not straight, but consists of several straight parts.

PROPOSITION X. THEOREM

979. *The area of a zone is equal to the product of its altitude by the circumference of a great circle.*



GIVEN—a zone formed by the revolution of the arc AD of the semicircle HAS about its diameter HS as an axis. Let $A'D'$ be the altitude of the zone and O the centre of the semicircle.

TO PROVE area zone $AD = A'D' \times 2\pi OA$.

Divide the arc AD into any number of equal parts, AB, BC, CD. Draw the chords AB, BC, CD.

Also draw AA', BB', CC', DD' perpendicular to HS and OM perpendicular to AB.

Denote by "area AB" the area of the surface generated by the straight line AB in revolving about HS.

Then area $AB = A'B' \times 2\pi OM$. § 977

Similarly area $BC = B'C' \times 2\pi OM$, §§ 164, 170

and area $CD = C'D' \times 2\pi OM$.

Adding these equations we have

area broken line $ABCD = (A'B' + B'C' + C'D') \times 2\pi OM$

$= A'D' \times 2\pi OM$.

Now let the number of divisions of the arc AD be increased indefinitely.

Then the broken line approaches the arc AD as a limit and OM approaches the radius OA of the sphere as a limit.

Moreover, the limit of the surface generated by the broken line $ABCD$ will be the surface generated by the limit of the broken line, that is, by the arc AD.

This latter is the zone AD.

Therefore area zone $AD = A'D' \times 2\pi OA$. § 186

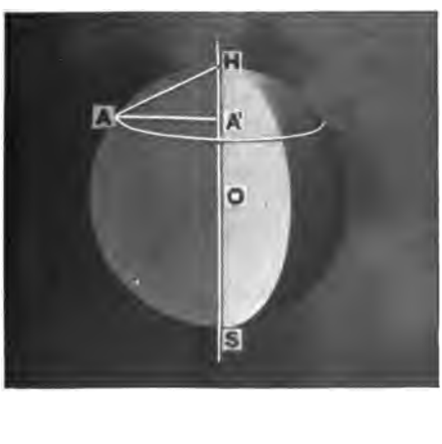
Q. E. D.

980. COR. I. Let S denote the area of the zone, H its altitude, and R the radius of the sphere.

Then $S = 2\pi RH$.

981. COR. II. *Two zones on the same sphere, or on equal spheres, are to each other as their altitudes.*

982. COR. III. *A zone of one base is equivalent to a circle whose radius is the chord of the generating arc of the zone.*



OUTLINE PROOF: Area zone $HA = 2\pi OH \times HA' = \pi HS \times HA' = \pi \overline{HA}^2$.

983. COR. IV. *The surface of a sphere is equivalent to four great circles.*

Hint.—The surface may be considered to be a zone whose altitude is the diameter of the sphere.

Hence its area is $2\pi R \times 2R = 4\pi R^2$.

984. COR. V. *Two spherical surfaces are to each other as the squares of their radii or as the squares of their diameters.*

PROPOSITION XI. LEMMA

985. *If a triangle revolve about an axis situated in its plane and passing through the vertex without crossing its surface, the volume generated will be equal to the area generated by the base multiplied by one-third of the altitude.*

GIVEN —the triangle ABC revolving about an axis XY passing through the vertex A without crossing the triangle. Let the altitude of the triangle be AD.

TO PROVE vol. gen. by ABC = area $BC \times \frac{1}{3}AD$.

FIG. 1

FIG. 2

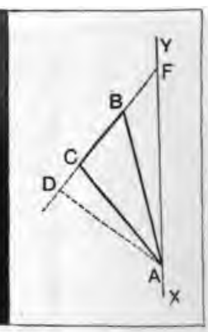


FIG. 3

CASE I. *When one side of the triangle ABC, as AB, lies in the axis.*

Draw CE perpendicular to the axis.

If this perpendicular falls within the triangle (Fig. 1), the volume generated by the triangle ABC is the sum of the volumes generated by the triangles BEC and AEC. That is, $\quad \text{vol. } ABC = \text{vol. } BEC + \text{vol. } AEC. \quad (1)$

If the perpendicular falls without the triangle (Fig. 2), the volume generated by the triangle ABC is the difference of the volumes generated by the triangles BEC and AEC. That is, $\quad \text{vol. } ABC = \text{vol. } BEC - \text{vol. } AEC. \quad (2)$

Now in either case

$$\text{vol. } BEC = \tfrac{1}{3}\pi \overline{EC}^2 \times BE \qquad \S\ 966$$

and

$$\text{vol. } AEC = \tfrac{1}{3}\pi \overline{EC}^2 \times AE.$$

Substituting these values in (1), we have

$$\text{vol. } ABC = \tfrac{1}{3}\pi \overline{EC}^2 \times (BE + AE).$$

For this case $\quad BE + AE = AB.$

Substituting in (2), we have

$$\text{vol. } ABC = \tfrac{1}{3}\pi \overline{EC}^2 \times (BE - AE).$$

For this case $\quad BE - AE = AB.$

Hence, in either case,

$$\text{vol. } ABC = \tfrac{1}{3}\pi \overline{EC}^2 \times AB$$
$$= \tfrac{1}{3}\pi EC \times EC \times AB.$$

But $\quad EC \times AB = BC \times AD,$

since each side is twice the area of the triangle ABC.

Therefore $\quad \text{vol. } ABC = \tfrac{1}{3}\pi EC \times BC \times AD.$

But $\pi EC \times BC$ is the area of the conical surface generated by BC. § 955

Therefore $\quad \text{vol. } ABC = \text{area } BC \times \tfrac{1}{3} AD.$ Q. E. D.

CASE II. *When the triangle ABC has neither side coinciding with the axis, and the base BC when produced meets the axis in F* (Fig. 3).

Then $\quad \text{vol. } ABC = \text{vol. } AFC - \text{vol. } AFB.$

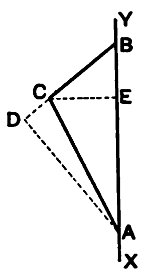


FIG. 1

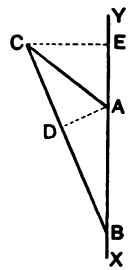


FIG. 2

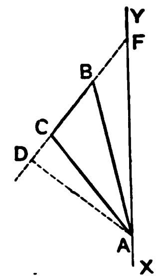


FIG. 3

But $\quad$ vol. AFC=area $FC \times \frac{1}{3}AD$,

and $\quad$ vol. AFB=area $FB \times \frac{1}{3}AD$. $\quad$ Case I

Therefore vol. ABC=(area FC−area FB)$\times \frac{1}{3}AD$

$\quad$ =area $BC \times \frac{1}{3}AD$. $\quad$ Q. E. D.

CASE III. *When the base BC of the triangle ABC is parallel to the axis.*

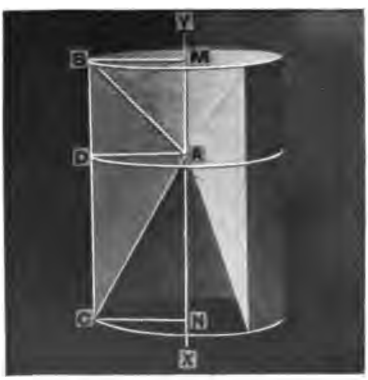


FIG. 4

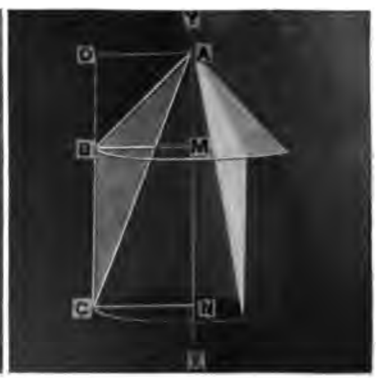


FIG. 5

According as AD falls within (Fig. 4) or without (Fig. 5) the triangle, we have

$$\text{vol. } ABC = \text{vol. } ADC + \text{vol. } ADB, \quad (3)$$

or

$$\text{vol. } ABC = \text{vol. } ADC - \text{vol. } ADB. \quad (4)$$

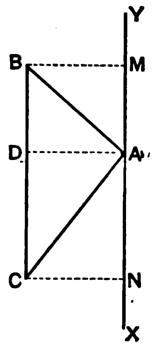


FIG. 4

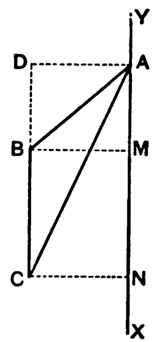


FIG. 5

Draw BM and CN perpendicular to XY.

Now for either figure

$$\begin{aligned} \text{vol. } ADC &= \text{vol. } ADCN - \text{vol. } ACN \\ &= \pi\overline{NC}^2 \times AN - \tfrac{1}{3}\pi\overline{NC}^2 \times AN \qquad \text{§§ 941, 966} \\ &= \tfrac{2}{3}\pi\overline{NC}^2 \times AN = \tfrac{2}{3}\pi AD^2 \times DC \\ &= 2\pi AD \times DC \times \tfrac{1}{3}AD. \end{aligned}$$

But $2\pi AD \times DC$ is the area of the cylindrical surface generated by DC. § 937

Therefore vol. ADC = area $DC \times \frac{1}{3}AD$.

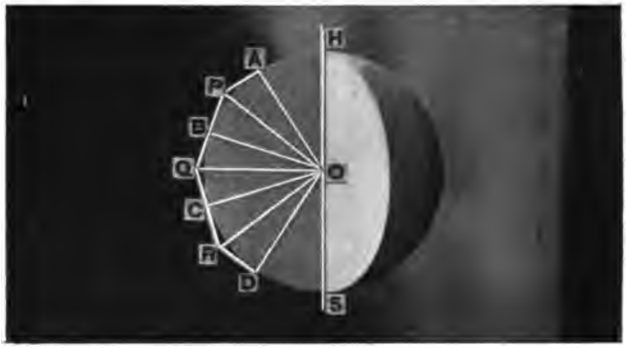
Similarly vol. ADB = area $DB \times \frac{1}{3}AD$.

Now, substituting these values for vol. ADC and vol. ADB in equations (3) and (4), and remembering that equation (3) applies to Fig. 4 and equation (4) to Fig. 5, we get

vol. ABC = area $BC \times \frac{1}{3}AD$. Q. E. D.

PROPOSITION XII. THEOREM

986. *The volume of a spherical sector is equal to the area of the zone which forms its base multiplied by one-third the radius of the sphere.*



GIVEN—a spherical sector, formed by the revolution of the sector AOD of the semicircle HAS about its diameter HS as an axis.

TO PROVE—vol. sph. sect. $AOD =$ area zone $AD \times \frac{1}{3}OA$.

Divide the arc AD into any number of equal parts, AB, BC, CD.

At A, B, C, and D draw tangents AP, PQ, QR, RD. Draw OB, OC, OP, OQ, OR.

The volume generated by the polygon $OAPQRD$ is the sum of the volumes generated by the triangles OAP, OPQ, OQR, ORD.

But vol. $AOP =$ area $AP \times \frac{1}{3}OA$ § 985

vol. $OPQ =$ area $PQ \times \frac{1}{3}OB =$ area $PQ \times \frac{1}{3}OA$

etc.

Hence

vol. $OAPQRD =$ (area $AP +$ area $PQ +$ etc.) $\times \frac{1}{3}OA$

$=$ area $APQRD \times \frac{1}{3}OA$.

Now let the number of divisions of the arc AD be indefinitely increased.

Then broken line $APQRD$ approaches arc AD as a limit;

surface generated by the broken line approaches surface generated by the arc as a limit;

that is, surface generated by the broken line approaches the zone AD as a limit;

volume generated by the polygon approaches volume generated by the sector;

that is, volume generated by the polygon approaches volume spherical sector AOD;
and OA is constant.

. Therefore vol. sph. sect. AOD = area zone $AD \times \frac{1}{3}OA$. § 186

Q. E. D.

987. COR. I. Let H denote the altitude of the zone which forms the base of the spherical sector.

Then $$\text{vol. sph. sector} = 2\pi RH \times \tfrac{1}{3}R$$
$$= \tfrac{2}{3}\pi R^2 H.$$

988. COR. II. *The volume of a sphere is equal to the area of its surface multiplied by one-third of its radius.*

Hint.—A sphere may be regarded as a spherical sector whose base is the surface of the sphere.

989. COR. III. *If V is the volume of a sphere, R its radius, and D its diameter,*

$$V = 4\pi R^2 \times \tfrac{1}{3}R = \tfrac{4}{3}\pi R^3 = \tfrac{1}{6}\pi D^3.$$

990. COR. IV. *The volumes of two spheres are to each other as the cubes of their radii, or as the cubes of their diameters.*

991. COR. V. *The volume of a spherical pyramid is equal to the area of its base multiplied by one-third the radius of the sphere.*

Hint.—Let v be the volume of the spherical pyramid, s the area of its base, and R the radius of the sphere.

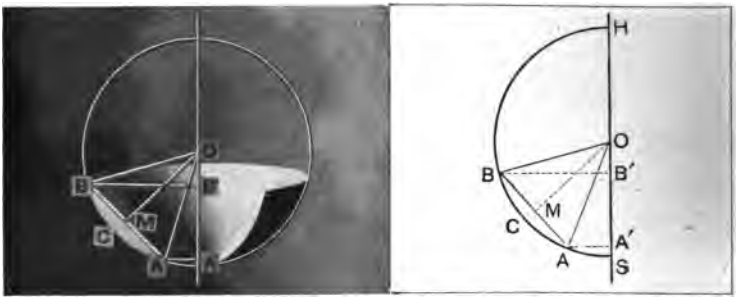
Also let V be the volume of the sphere and S the area of its surface.

Then $$\frac{v}{V} = \frac{s}{S}.$$ § 899 XII

And $$V = S \times \tfrac{1}{3}R.$$

PROPOSITION XIII. THEOREM

992. *The volume of the solid generated by a circular segment revolving about a diameter exterior to it is equal to one-sixth the area of the circle whose radius is the chord of the segment multiplied by the projection of that chord upon the axis.*



GIVEN—a circular segment ACB revolving about the diameter HS. Let $A'B'$ be the projection of AB upon HS.

TO PROVE $\quad$ vol. $ACB = \frac{1}{6}\pi\overline{AB}^2 \times A'B'$.

Draw the radii OA, OB, and draw OM perpendicular to AB.

Then vol. ACB = vol. sector AOB − vol. triangle AOB.

Now $\quad$ vol. sector AOB = zone $ACB \times \frac{1}{3}OA$ $\quad$ § 986

$= 2\pi OA \times A'B' \times \frac{1}{3}OA$ $\quad$ § 980

$= \frac{2}{3}\pi\overline{OA}^2 \times A'B'$,

and $\quad$ vol. triangle AOB = area $AMB \times \frac{1}{3}OM$ $\quad$ § 985

$= 2\pi OM \times A'B' \times \frac{1}{3}OM$ $\quad$ § 977

$= \frac{2}{3}\pi\overline{OM}^2 \times A'B'$.

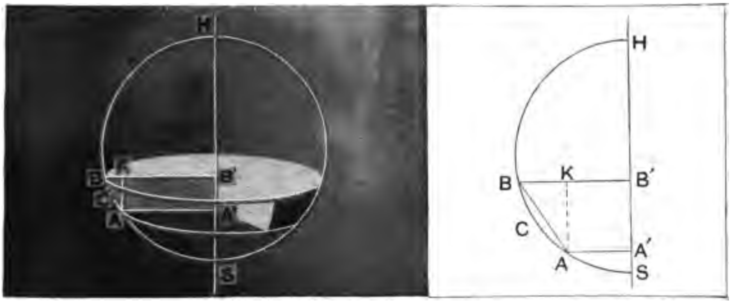
Hence $\quad$ vol. $ACB = \frac{2}{3}\pi(\overline{OA}^2 - \overline{OM}^2) \times A'B'$.

But $\quad$ $\overline{OA}^2 - \overline{OM}^2 = \overline{AM}^2 = \frac{1}{4}\overline{AB}^2$. $\quad$ §§ 318, 167

Therefore $\quad$ vol. $ACB = \frac{1}{6}\pi\overline{AB}^2 \times A'B'$. $\quad$ Q. E. D.

PROPOSITION XIV. THEOREM

993. *The volume of a spherical segment is equal to half the sum of its bases multiplied by its altitude increased by the volume of a sphere whose diameter is equal to that altitude.*



GIVEN—a spherical segment, generated by the revolution of the figure $ACBB'A'$ about the diameter HS of the semicircle HBS, the lines AA' and BB' generating the bases, and the arc ACB generating the curved surface of the segment. Denote BB' by r, AA' by r', $A'B'$ by h, and the volume of the spherical segment by V.

TO PROVE $V=\frac{1}{2}(\pi r^2+\pi r'^2)h+\frac{1}{6}\pi h^3.$

The volume of the spherical segment is the sum of the volume generated by the circular segment ACB and the volume of the frustum of a cone generated by the trapezoid $ABB'A'$.

Hence $V=\frac{1}{6}\pi\overline{AB}^2\times h+\frac{1}{3}\pi(r^2+r'^2+rr')h.$ (1) §§ 992, 970

Draw AK perpendicular to BB'.

Then $BK=r-r'.$

Hence $\overline{BK}^2=r^2+r'^2-2rr'.$

Now $\overline{AB}^2=\overline{AK}^2+\overline{BK}^2=h^2+r^2+r'^2-2rr'.$ § 317

Substituting this value for $\overline{AB}^2$ in (1), we get

$$V=\frac{1}{2}(\pi r^2+\pi r'^2)h+\frac{1}{6}\pi h^3.$$ Q. E. D.

994. COR. *The formula for the volume of a spherical segment of one base is*

$$V=\tfrac{1}{2}\pi r^2 h+\tfrac{1}{6}\pi h^3.$$

Hint.—This is obtained from the preceding formula by making the radius r' of one base equal to zero.

PROBLEMS OF DEMONSTRATION

995. *Exercise.*—The lateral area of a cylinder of revolution is equal to the area of a circle the radius of which is a mean proportional between the altitude of the cylinder and the diameter of its base.

996. *Exercise.*—The volume of a cylinder is equal to the product of the area of a right section by an element.

997. *Exercise.*—The area of a sphere is equal to the lateral area of a circumscribed cylinder of revolution.

998. *Exercise.*—The volume of a sphere is two-thirds the volume of a circumscribed cylinder of revolution.

999. *Exercise.*—If a cylinder of revolution of which the altitude is equal to the diameter of the base, and a cone of

revolution of which the slant height is equal to the diameter of the base, be inscribed in a sphere; the total area of the cylinder is a mean proportional between the area of the sphere and the total area of the cone, and the volume of the cylinder is a mean proportional between the volume of the sphere and the volume of the cone.

1000. *Exercise.*—If a cylinder of revolution of which the altitude is equal to the diameter of the base, and a cone of revolution of which the slant height is equal to the diameter of the base, be circumscribed about a sphere; the total area of the cylinder is a mean proportional between the area of the sphere and the total area of the cone, and the volume of the cylinder is a mean proportional between the volume of the sphere and the volume of the cone.

1001. *Exercise.*—Show that two cylinders of revolution, whose lateral areas are equal, are to each other as their radii, or inversely as their altitudes.

PROBLEMS FOR COMPUTATION

1002. (1.) A right section of a cylinder is a circle whose radius is 3 ft.; an element of the cylinder is 13 ft. Find the lateral area.

(2.) A cylindrical boiler is 12 ft. long and 6 ft. in diameter. Find its surface, and the number of gallons of water it will hold.

(3.) A cylindrical pail is 6 in. deep and 7 in. in diameter. Find its contents and the amount of tin required for its construction.

(4.) Find the volume generated by a rectangle 9 dcm. long and 4 dcm. broad (*a*) in revolving about its longer side; (*b*) in revolving about its shorter side.

(5.) A conical cistern is 13 in. deep and 12 in. across the top, which is circular. Find its contents.

(6.) A conical church steeple is 50 ft. high and 10 ft. in diameter at the base. How much would it cost to paint the steeple at 10 cents a square foot?

(7.) A cube, an edge of which is 2 in., is inscribed in a cone of revolution, of which the altitude is 5 in. Find the volume of the cone.

(8.) The sides of an equilateral triangle are each 10 in. What is the volume generated, if the triangle revolve about its altitude? What is the area of the surface generated?

(9.) The sides of a triangle are each 12.49 in. What is the volume generated if the triangle revolve about one side? What is the area of the surface generated?

(10.) Find the total area of a frustum of a cone of revolution, the radii of whose bases are 12 cm. and 7 cm., and whose altitude is 9.7 cm.

(11.) Find the volume of a frustum of a circular cone, the areas of whose bases are 12 sq. in. and 8 sq. in., and whose altitude is 5 in.

(12.) If the radius of a sphere is 647 cm.,

(*a.*) What is its area?

(*b.*) What is its volume?

(*c.*) What is the area of a lune whose angle is 35°?

(*d.*) What is the volume of the spherical ungula whose base is the preceding lune?

(*e.*) What is the area of a spherical polygon whose angles are 140°, 65°, 120°, 50°?

(*f.*) What is the volume of the spherical pyramid whose base is the preceding spherical polygon.

(13.) A sphere and a cylinder of revolution have equal

areas. What is the ratio of the area of a sphere of half the diameter to the area of a similar cylinder of two-thirds the altitude?

(14.) How many marbles $\frac{3}{4}$ in. in diameter can be made from 100 cu. in. of glass, if there is no waste in melting?

(15.) Assuming the earth to be a sphere 7960 miles in diameter, what is the area of its surface? What is its volume?

(16.) The surface of a sphere is 1.514 sq. m. Find its radius.

(17.) The volume of a sphere is 1000 cu. in. Find its radius.

(18.) The surface of a sphere is 4632 sq. m. Find its volume.

(19.) Show that, if S is the surface of a sphere and V its volume, $$36\pi V^2 = S^3.$$

(20.) A hollow rubber ball is 2 in. in diameter and the rubber is $\frac{3}{16}$ in. thick. How much rubber would be used in the manufacture of 1000 such balls?

(21.) If a sphere of iron weighs 999 lbs., how much would a sphere of iron of one third the diameter weigh?

(22.) A cone of revolution and a cylinder of revolution each have as base a great circle of a sphere, and as altitude the radius of the sphere. Find the ratios of the total surfaces of the cone and cylinder to the surface of the sphere.

(23.) Find the ratio of the volumes of a cone of revolution and a cylinder of revolution to the volume of a sphere, if the bases of the cone and cylinder are each equal to a great circle of the sphere, and the altitudes of the cone and cylinder are each equal to the diameter of the sphere.

(24.) Find the total area and the volume of the cylinder and of the cone in Problem (22), if the radius of the sphere is 1 dcm.

(25.) Find the total area and the volume of the cylinder and of the cone in Problem (23), if the radius of the sphere is 59.77 cm.

(26.) If the radius of a sphere is 4.581 in., what is the area of a zone whose altitude is 1.456 in.?

(27.) A dome is in the form of a spherical zone of one base, and its height is 30 ft. Find its surface if the radius of the sphere is 35 ft.

(28.) The radius of a sphere is 6.742 in.; the altitude of a zone is 2 in. Find the volume of the spherical sector of which this zone is the base.

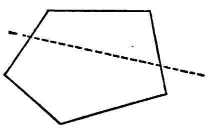
(29.) Find the volume of a spherical segment of one base whose altitude is 3 cm., and the radius of whose base is 9.643 cm.

PLANE GEOMETRY

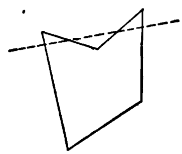
APPENDIX

1003. *Def.*—A polygon is **convex**, if no straight line can meet its perimeter in more than two points. [Repeated from § 65.]

1004. *Def.*—A polygon is **re-entrant**, if a straight line can be drawn meeting its perimeter in more than two points.



CONVEX POLYGON



RE-ENTRANT POLYGON

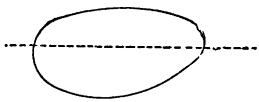
1005. *Def.*—A **curved line**, or **curve**, is a line, no part of which is straight. [Repeated from § 765.]

CURVED LINES

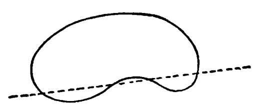
1006. *Def.*—A curve is **closed**, if it returns upon itself.

1007. *Def.*—A closed curve is **convex**, if no straight line can meet it in more than two points.

1008. *Def.*—A closed curve is **re-entrant**, if a straight line can be drawn meeting it in more than two points.



CONVEX CLOSED CURVE



RE-ENTRANT CLOSED CURVE

15

PROPOSITION I. THEOREM

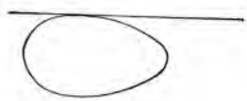
1009. *The circumference of a circle is a convex curve.*

For, if a straight line could meet it in three points, we would have three equal straight lines (the radii to the points of intersection) from a point to a straight line. It follows from § 100 that this is impossible.

1010. *Def.*—A **secant** of a convex closed curve is a straight line meeting it in two points.

1011. *Def.*—A **tangent** to a convex closed curve is a straight line meeting it in only one point, however far the line is produced.

SECANT



TANGENT

PROPOSITION II. THEOREM

1012. *A tangent can be drawn at any point of a convex closed curve.*

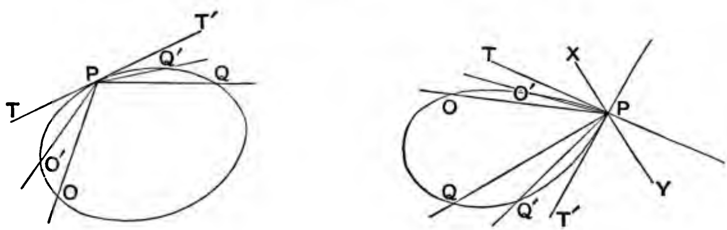


FIG. 1 FIG. 2

If we imagine a secant PO of a convex closed curve to revolve about one point of intersection P as a pivot, while the intersection O moves along the curve toward P, the limiting position of this secant, when O coincides with P, will be a tangent PT to the curve at P.

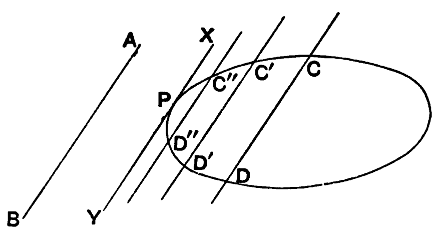
Q. E. D.

1013. *Remark.*—If we suppose a secant PQ to revolve in the opposite direction, so that Q approaches P on the other side, its limiting position will in general be another tangent PT' at the point P (Fig. 2). In this case there will be an infinite number of other tangents, as XY. These will all lie in the angle through which PT must turn, in the direction in which PO revolved, to coincide with PT'. In what follows we will suppose that the two tangents, PT and PT', are the same straight line, as in Fig. 1, thus excluding from consideration curves like that in Fig. 2.

That is, *to the curves here considered, only one tangent can be drawn at a given point.*

PROPOSITION III. THEOREM

1014. *Two tangents can be drawn to a convex closed curve parallel to a given straight line.*



Draw a secant CD parallel to the given straight line AB.

Suppose CD to move in a direction perpendicular to AB, but always remaining parallel to AB.

The points C and D will move along the curve and will ultimately come together so as to coincide.

If this limiting position of CD is XY, P being the point in which C and D coincide, then XY is tangent to the curve at P and is parallel to AB.

If CD had moved in the opposite direction we would have obtained another tangent to the curve parallel to AB. Hence we see that two tangents can be drawn to a convex closed curve parallel to a given straight line. Q. E. D.

PROPOSITION IV. THEOREM

1015. *A convex closed curve is greater than the perimeter of any inscribed polygon.*

Hint.—The proof is identical with that of Proposition V., Book V.

PROPOSITION V. THEOREM

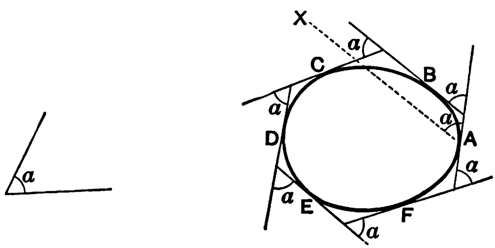
1016. *A convex closed curve is less than the perimeter of a circumscribed polygon or any enveloping line.*

Hint.—The proof is identical with that of Proposition VI., Book V.

1017. *Def.*—A polygon is **equiangular** when its angles are all equal.

PROPOSITION VI. LEMMA

1018. *About any given convex closed curve an equiangular polygon of any required number of sides can be circumscribed.*



GIVEN $ABCDEF$, any convex closed curve.

TO PROVE—an equiangular polygon of n sides can be circumscribed.

Take the angle a one-n^{th} of four right angles.

At any point A on the circumference draw a tangent, and from A draw the secant AX, making the angle a with the tangent.

Draw a tangent to the curve parallel to AX, and let the point of tangency be B.

Then the tangent at B makes the angle a with the tangent at A. § 49

In like manner draw a tangent at C, making the angle a with the second tangent, and so proceed until n tangents are drawn.

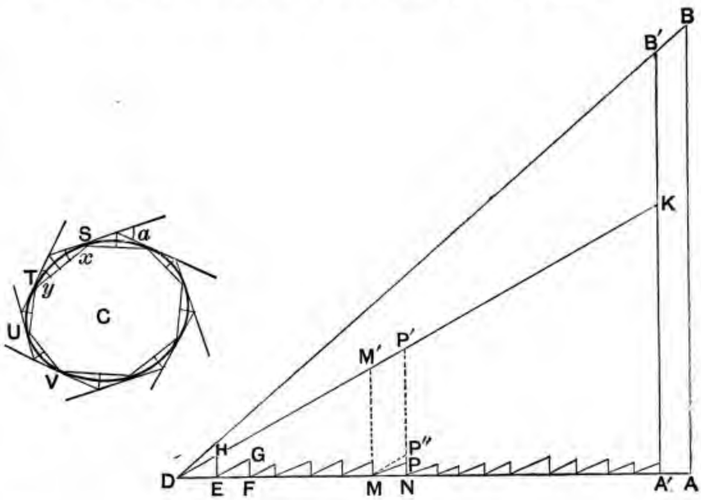
The last tangent must form in like manner the angle a with the first tangent, otherwise the sum of the n exterior angles of the polygon would not equal four right angles. § 69

Since the exterior angles are all equal, their supplementary angles, the angles of the polygon, must be all equal.

Hence the polygon is equiangular and of n sides. Q. E. D.

PROPOSITION VII. THEOREM

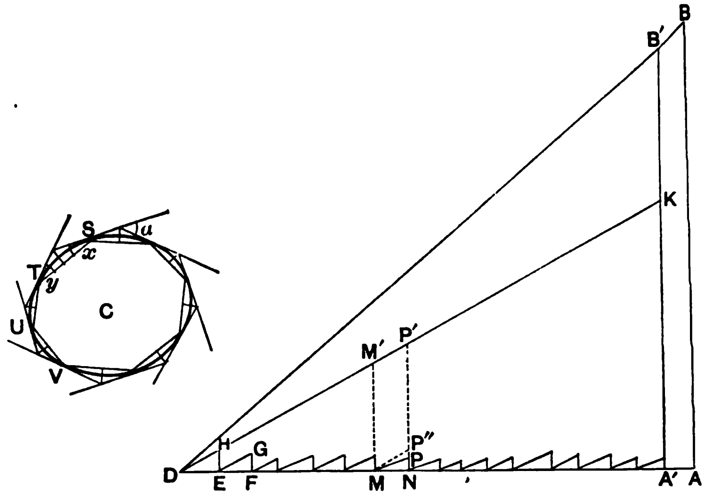
1019. *The circumference of a convex closed curve is the limit which the perimeters of a series of inscribed and circumscribed polygons approach when the number of their sides is indefinitely increased; and the area of the curve is the limit of the areas of these polygons.*



GIVEN any convex closed curve C.

TO PROVE—I. Its circumference is the common limit which the perimeters of a series of inscribed and circumscribed polygons approach when the number of their sides is indefinitely increased.

II. The area of the curve is the common limit which the areas of the inscribed and circumscribed polygons approach when the number of their sides is indefinitely increased.



Draw AB of any assigned length, no matter how small. We shall prove that the difference between the inscribed and circumscribed perimeters can be made less than AB.

Conceive the straight line AD, equal in length to the circumference of C, to be drawn perpendicular to AB. Join BD.

Now divide 4 right angles into such a number of equal parts that each part a shall be less than the angle BDA. Let n be the number of parts.

Circumscribe about C an equiangular polygon of n sides whose exterior angle is a. § 1018

Join the points of tangency, forming an inscribed polygon of n sides.

From the vertices of the circumscribed polygon draw perpendiculars to the sides of inscribed polygon, thus forming $2n$ right triangles.

The angles at S, T, U, V, etc., between the tangents and chords, as x or y, are each less than a (§ 59), and therefore still less than angle ADB.

Of these angles x, y, etc., select the greatest, and place the right triangle containing it within the angle ADB in the position DEH.

In like manner place all the other right triangles along DA, as EFG, etc., irrespective of order. Let the last one extend to A'.

Thus the sum of the bases, or DA', equals the inscribed perimeter (less than DA), and the sum of the hypotenuses equals the circumscribed perimeter.

Produce DH to meet the perpendicular $A'B'$ at K.

I. Any hypotenuse, as MP, can easily be proved less than $M'P'$, the portion of DK included between perpendiculars at M and N. § 99

Hence, adding all such inequalities, DK is greater than the sum of the hypotenuses DH, EG, etc.

That is, DK is greater than the circumscribed perimeter.

Now DA' is equal to the inscribed perimeter.

Hence the difference of the circumscribed and inscribed perimeters is less than $DK - DA'$.

But $DK - DA'$ is less than KA'. § 137

And KA' is less than $A'B'$. And $A'B'$ is less than AB.

Much more, therefore, is the difference of the perimeters less than AB.

We can thus make the difference of perimeters as small as we please.

But the circumference is always intermediate between the perimeters.

Hence either perimeter can be made to differ from the circumference by less than any assigned quantity.

Therefore the circumference is the common limit to which the perimeters approach. Q. E. D.

II. Moreover, the difference between the areas of the inscribed and circumscribed polygons consists of the $2n$ right triangles, which is less than the triangle ADB.

But since the base AD of this triangle is constant we can make its area as small as we please by making its altitude AB as small as we please. § 187

Hence the difference between the polygons can be made as small as we please.

But the area of the curve is always intermediate between the polygons.

Hence either polygon can be made to differ from the area of the curve by less than any assigned quantity.

Therefore the area of the curve is the common limit to which the polygon-areas approach. Q. E. D.

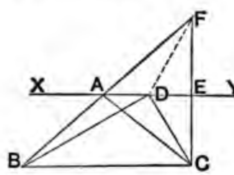
MAXIMA AND MINIMA OF PLANE FIGURES

1020. *Def.*—Of the values which a variable quantity assumes, the largest value is called **the maximum**; the smallest, **the minimum.**

Thus, the diameter of a circle is the maximum among all straight lines joining two points of the circumference; and among all the lines drawn from a given point to a given straight line the perpendicular is the minimum.

PROPOSITION VIII. THEOREM

1021. ***Of all triangles having the same base and equal areas, that which is isosceles has the minimum perimeter.***



GIVEN—the isosceles triangle ABC and *any* other triangle DBC having an equal area and the same base BC.

TO PROVE—the perimeter of ABC is less than the perimeter of DBC.

Outline proof.—The vertices A and D are in the straight line XY parallel to BC. (Why?)

Draw CE perpendicular to BC, meeting BA produced at F. Join DF.

The angle CAE=angle FAE, and the triangle CAE=triangle FAE.

Hence AE is perpendicular to CF at its middle point.

Now $$AB+AF<DB+DF.$$

Or $$AB+AC<DB+DC.$$

Hence $$BC+AB+AC<BC+DB+DC.$$ Q. E. D.

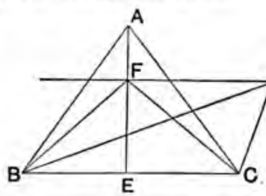
1022. *Remark.*—The converse of the preceding proposition is also true, viz.: *Of all triangles having the same base and equal areas, that which has the minimum perimeter is isosceles.* In fact, it is practically the same theorem as the proposition itself, for there is only one isosceles triangle fulfilling the given conditions, and only one triangle of minimum perimeter fulfilling the given conditions; just as to say that John Smith is the tallest man in the room is equivalent to saying that the tallest man in the room is John Smith, provided we know that there is only one John Smith in the room and only one tallest man.

1023. COR. *Of all triangles having the same area, that which is equilateral has the minimum perimeter.*

1024. *Def.*—When two figures have equal perimeters they are called **isoperimetric.**

PROPOSITION IX. THEOREM

1025. *Of all isoperimetric triangles having the same base, that which is isosceles has the maximum area.*



GIVEN—the isosceles triangle ABC and *any* other triangle DBC having an equal perimeter and the same base BC.

TO PROVE the area of $ABC >$ area DBC.

Outline proof.—Draw AE perpendicular to BC and DF parallel to BC.

Join FB and FC.

The triangles FBC and DBC have equal areas.

But FBC is isosceles.

Therefore perimeter $FBC <$ perimeter DBC.

Or perimeter $FBC <$ perimeter ABC.

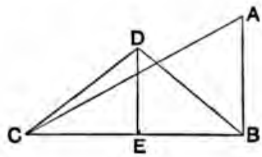
Hence $BF < BA$, and $FE < AE$.

Therefore the area of triangle $ABC >$ area of triangle DBC. Q. E. D.

1026. COR. *Of all isoperimetric triangles, that which is equilateral has the maximum area.*

PROPOSITION X. THEOREM

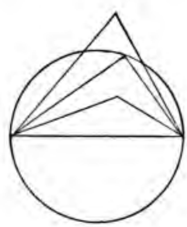
1027. *Of all triangles having two sides of one equal to two sides of the other, that in which these two sides are perpendicular to each other is the maximum.*



The proof is left to the student.

PROPOSITION XI. THEOREM

1028. *The locus of the vertex of a right angle whose sides pass through two fixed points is the circumference of a circle whose diameter is the straight line joining those points.*



Hint.—Apply §§ 202, 207, 210.

PROPOSITION XII. THEOREM

1029. *Of all isoperimetric plane figures, the maximum figure is a circle.*

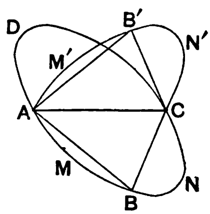


FIG. 1

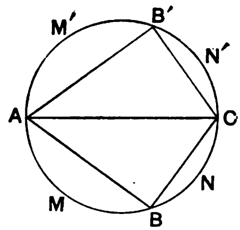


FIG. 2

GIVEN—among any number of isoperimetric plane figures the figure of maximum area *ABCDA*.

TO PROVE that *ABCDA* is a circle.

First, draw any straight line *AC* (Fig. 1), dividing its perimeter into two parts of equal length.

Then *AC* will also divide its surface into two parts of equal area.

For, if not, as for example if area $ABC >$ area ADC, form the figure $AB'C$ symmetrical to *ABC* by revolving *ABC* on *AC* as an axis.

Then the figure $ABCB'A$ would be greater than *ABCDA* and would have the same perimeter.

Hence *ABCDA* would not be the maximum.

This would be contrary to the hypothesis.

Therefore *AC* does divide the surface into two equivalent parts.

Secondly, take any point *B* in the semiperimeter *ABC*.

We will prove that the angle *ABC* is a right angle.

Form the figure $AB'C$ symmetrical to *ABC*.

Then area $AB'C =$ area ABC.

But we have just proved area $ABC =$ area ADC.

Therefore area $ABCB'A =$ area $ABCDA$. Ax. 2

That is, $ABCB'A$ is equivalent to the maximum figure.

Now, if the angle *ABC* were not a right angle, we could increase the area of the equal triangles *ABC* and $AB'C$ by moving the points *A* and *C* nearer together or farther apart, so as to make *ABC* a right angle, without changing the lengths of the straight lines *AB*, *BC*, AB', $B'C$, and without changing the areas of the segments *AMB*, *BNC*, $AM'B'$, $B'N'C$.

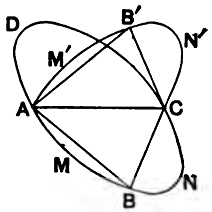


FIG. 1

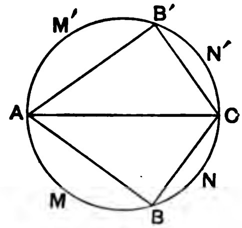


FIG. 2

In doing this the area of the whole figure $ABCB'A$ would be increased.

But this is impossible, since $ABCB'A$ is equivalent to the maximum figure.

Therefore ABC must be a right angle.

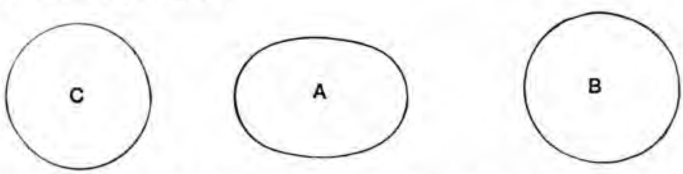
In like manner, if we should choose any point D in the semiperimeter ADC, we could show that ADC would be a right angle.

Therefore the figure $ABCDA$ is a circle. § 1028

Q. E. D.

PROPOSITION XIII. THEOREM

1030. *Of all plane figures containing the same area, the circle has the minimum perimeter.*



GIVEN—a circle C, and any other figure A having the same area as C.

TO PROVE the perimeter of C is less than that of A.

Let B be a circle having the same perimeter as the figure A.

Then area $A <$ area B, or area $C <$ area B. § 1029

Now, of two circles, that which is the less has the less perimeter. §§ 491, 499

Therefore the perimeter of C is less than the perimeter of B, or less than the perimeter of A. Q. E. D.

PROPOSITION XIV. THEOREM

1031. ***Of all the polygons constructed with the same given sides, that is the maximum which can be inscribed in a circle.***

GIVEN—a polygon P inscribed in a circle, and P', any other polygon constructed with the same sides and not inscriptible in a circle.

TO PROVE that $P>P'$.

Upon the sides of the polygon P' construct circular segments equal to those on the corresponding sides of P.

The whole figure S' thus formed has the same perimeter as the circle S.

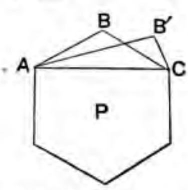
Therefore area of $S>$area of S'. § 1029

Subtracting the circular segments from both,

$$P>P'.$$ Q. E. D.

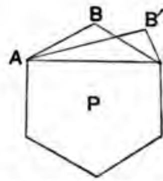
PROPOSITION XV. THEOREM

1032. *Of all isoperimetric polygons having the same number of sides the maximum is a regular polygon.*



GIVEN—P the maximum of all the isoperimetric polygons of the same number of sides.

TO PROVE that P is a regular polygon.



If two of its sides, as AB', $B'C$, were unequal, the isosceles triangle ABC, having the same perimeter as $AB'C$ and a greater area, could be substituted for the triangle ABC. § 1025

This would increase the area of the whole polygon without changing the length of the perimeter or the number of its sides.

Hence the sides of the maximum polygon must be all equal.

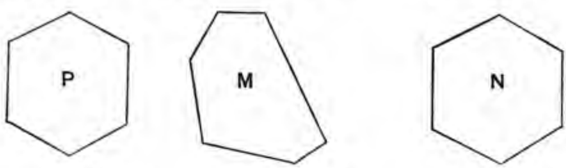
But the maximum of all polygons constructed with the same given sides must be inscriptible in a circle. § 1031

Therefore P is a regular polygon. §§ 164, 469

Q. E. D.

PROPOSITION XVI. THEOREM

1033. *Of all polygons having the same number of sides and the same area, the regular polygon has the minimum perimeter.*



GIVEN—P, a regular polygon, and M, any other polygon of the same number of sides and area as P.

TO PROVE the perimeter of $P <$ perimeter of M.

Let N be a regular polygon having same perimeter and same number of sides as M.

Then area $M <$ area N, or area $P <$ area N. § 1032

But of two regular polygons of the same number of sides the one of less area has the less perimeter. §§ 482, 483

Therefore the perimeter P is less than that of N, or less than that of M.

Q. E. D.

EXERCISES

BOOK I

PROBLEMS OF DEMONSTRATION

1. The bisector of an angle of a triangle is less than half the sum of the sides containing the angle.

2. The median drawn to any side of a triangle is less than half the sum of the other two sides, and greater than the excess of that half sum above half the third side.

3. The shortest of the medians of a triangle is the one drawn to the longest side.

4. The sum of the three medians of a triangle is less than the sum of the three sides, but greater than half their sum.

5. In any triangle the angle between the bisector of the angle opposite any side and the perpendicular from the opposite vertex on that side is equal to half the difference of the angles adjacent to that side.

6. LM and PR are two parallels which are cut obliquely by AB in the points A, B, and at right angles by AC in the points A, C; the line BED, which cuts AC in E and LM in D, is such that ED is equal to $2AB$. Prove that the angle DBC is one-third the angle ABC.

7. The sum of the diagonals of a quadrilateral is less than the sum of the four lines joining any point other than the intersection of the diagonals to the four vertics.

8. The difference between the acute angles of a right triangle is equal to the angle between the median and the perpendicular drawn from the vertex of the right angle to the hypotenuse.

9. In a right triangle the bisector of the right angle also bisects the angle between the perpendicular and the median from the vertex of the right angle to the hypotenuse.

10. In the triangle formed by the bisectors of the exterior angles of a given triangle, each angle is one-half the supplement of the opposite angle in the given triangle.

11. A right triangle can be divided into two isosceles triangles.

12. A median of a triangle is greater than, equal to, or less than half of the side which it bisects, according as the angle opposite that side is acute, right, or obtuse.

13. The point of intersection of the perpendiculars erected at the middle of each side of a triangle, the point of intersection of the three medians, and the point of intersection of the three perpendiculars from the vertices to the opposite sides are in a straight line; and the distance of the first point from the second is half the distance of the second from the third.

14. Find the locus of a point the sum or the difference of whose distances from two fixed straight lines is given.

15. On the side AB, produced if necessary, of a triangle ABC, AC' is taken equal to AC; similarly on AC, AB' is taken equal to AB, and the line $B'C'$ drawn to cut BC in P. Prove that the line AP bisects the angle BAC.

16. The point of intersection of the straight lines which join the middle points of opposite sides of a quadrilateral is the middle point of the straight line joining the middle points of the diagonals.

17. The angle between the bisector of an angle of a triangle and the bisector of an exterior angle at another vertex is equal to half the third angle of the triangle.

18. If L and M are the middle points of the sides AB, CD of a parallelogram $ABCD$, the straight lines, DL, BM trisect the diagonal AC.

19. ABC is an equilateral triangle; BD and CD are the bisectors of the angles at B and C. Prove that lines through D parallel to the sides AB and AC trisect BC.

20. The angle between the bisectors (produced only to their point of intersection) of two adjacent angles of a quadrilateral is equal to half the sum of the two other angles of the quadrilateral. The acute angle between the bisectors of two opposite angles of a quadrilateral is equal to half the difference of the other angles.

21. The bisectors of the angles of a quadrilateral form a second quadrilateral of which the opposite angles are supplementary. When the first quadrilateral is a parallelogram, the second is a rectangle whose diagonals are parallel to the sides of the parallelogram and each equal to the difference of two adjacent sides of the parallelogram. When the first quadrilateral is a rectangle, the second is a square.

22. Two quadrilaterals are equal if an angle of the one is equal to an angle of the other, and the four sides of the one are respectively equal to the four similarly situated sides of the other.

23. If two polygons have the same number of sides and this number is odd, and if one polygon can be placed upon the other so that the middle points of the sides of the first fall upon the middle points of the sides of the second, the polygons are equal.

PROBLEMS OF CONSTRUCTION

24. Find a point in a straight line such that the sum of its distances from two fixed points on the same side of the straight line shall be the least possible.

25. Find a point in a straight line such that the difference of its distances from two fixed points on opposite sides of the line shall be the greatest possible.

26. Draw through a given point within a given angle a straight line such that the part intercepted between the sides of the angle shall be bisected by the given point.

27. Through a given point without a straight line to draw a straight line making a given angle with the given line.

28. Divide a rectangle 7 in. long and 3 in. broad into three figures which can be joined together so as to form a square.

BOOK II

PROBLEMS OF DEMONSTRATION

29. If a circle is circumscribed about an equilateral triangle and from any point in the circumference straight lines are drawn to the three vertices, one of these lines is equal to the sum of the other two.

30. If one circle touches another internally at P and a tangent to the first at Q intersects the second in M, N, then the angles MPQ, NPQ are equal.

31. The centre of one circle is on the circumference of another; if A and B are the points in which the common tangents touch the second, prove that the line AB is tangent to the first.

32. The trapezoid of which the non-parallel sides are equal is the only trapezoid which can be inscribed in a circle.

33. From any point on the circumference of a circle circumscribed about an equilateral triangle ABC, straight lines are drawn parallel respectively to BC, CA, AB, meeting the sides CA, AB, BC at M, N, O. Prove that M, N, O are in the same straight line.

34. If a quadrilateral be inscribed in a circle and the opposite sides produced to meet at M and N, prove that the bisectors of the angles at M and N meet at right angles.

35. Two circles pass through the vertex and a point in the bisector of an angle. Prove that the portions of the sides of the angle intercepted between their circumferences are equal.

36. Each angle formed by joining the feet of the perpendiculars of a triangle is bisected by the perpendicular from the opposite vertex.

37. Circumscribe a circle about a triangle; from one vertex drop a perpendicular on the opposite side to meet it in M, and produce to meet the circumference in N. Then, if P is the intersection of the perpendiculars, $PM = MN$.

38. A fixed circle touches a fixed straight line; any circle is drawn touching the fixed circle at B and the fixed straight line at C. Prove that the straight line BC passes through a fixed point.

39. The distance from the centre of the circle circumscribed about a triangle to a side is equal to half the distance from the opposite vertex to the intersection of the three perpendiculars from the vertices to the sides.

40. Prove that the straight lines joining the vertices of a triangle with the opposite points of tangency of the inscribed circle meet in a point.

41. If two points are given on the circumference of a given circle, another fixed circle can be found such that if any two lines be drawn from the given points to intersect on its circumference, the straight line joining the points in which these lines meet the given circle a second time will be of constant length.

42. If the three diagonals joining the opposite vertices of a hexagon are equal and the opposite sides are parallel in pairs, the hexagon can be inscribed in a circle.

43. Equilateral triangles are constructed on the sides of a given triangle and external to it. Prove that the three lines, each joining the outer vertex of one of the equilateral triangles to the opposite vertex of the given triangle, meet in a point and are equal.

44. On each side of a triangle construct an isosceles triangle with the adjacent angles equal to 30°. Prove that the straight lines joining the outer vertices of these three triangles are equal.

LOCI

45. One side and the opposite angle of a triangle are given, and equilateral triangles are constructed on the other two (variable) sides. Find the locus of the middle point of the straight line joining the outer vertices of the equilateral triangles.

46. Through a vertex of an equilateral triangle is drawn any straight line PQ, terminated by the perpendiculars to the opposite side erected at the extremities of that side; on PQ as a side a second equilateral triangle is constructed. Find the locus of the opposite vertex of the second equilateral triangle.

47. The sides of a right triangle are given in position, its hypotenuse in length. Find the locus of the middle point of the hypotenuse.

48. AC, BD, are fixed diameters of a circle, at right angles to each other, and P is any point on the circumference. PA cuts BD in E; EF, parallel to AC, cuts PB in F. Prove that the locus of F is a straight line.

PROBLEMS OF CONSTRUCTION

49. Draw four circles through a given point and tangent to two given circles.

50. Through a given point draw a straight line cutting a given straight line and a given circle, such that the part of the line between the point and the given line may be equal to the part within the given circle.

51. Find a point in a given straight line such that tangents from it to two given circles shall be equal.

52. Construct a right triangle, having given one side and the perpendicular from the vertex of the right angle on the hypotenuse.

53. The distances from a point to the three nearest corners of a square are 1 in., 2 in., $2\frac{1}{2}$ in. Construct the square.

54. Construct a right triangle, having given the medians from the extremities of the hypotenuse.

55. Construct a right triangle, having given the difference between the hypotenuse and each side.

56. Construct a triangle, having given one angle and the medians drawn from the vertices of the other angles.

57. Construct a triangle, having given an angle, the perpendicular from its vertex on the opposite side, and the sum of the sides including that angle.

58. Having given two concentric circles, draw a chord of the larger circle, which shall be divided into three equal parts by the circumference of the smaller circle.

59. Inscribe in a circle a quadrilateral $ABCD$, having the diagonal AC given in direction, the diagonal BD in magnitude, and having given the position of the point E in which the sides AB and CD meet when produced.

60. Draw a chord of given length through a given point, within or without a given circle.

61. Construct an equilateral triangle such that one vertex is at a given point, and the other two vertices are on a given straight line and a given circumference respectively.

BOOK III

PROBLEMS OF DEMONSTRATION

62. If from a given point straight lines are drawn to the extremities of any diameter of a given circle, the sum of the squares of these lines will be constant.

63. The straight line joining the middle of the base of a triangle to the middle point of the line drawn from the opposite vertex to the point at which the inscribed circle touches the base, passes through the centre of the inscribed circle.

64. The square of the straight line joining the centre of a circle to to any point of a chord plus the product of the segments of the chord is equal to the square of the radius.

65. P and Q are two points on the circumscribing circle of the triangle ABC, such that the distance of either point from A is a mean proportional between its distances from B and C. Prove that the difference between the angles PAB, QAC is half the difference between the angles ABC, ACB.

66. If a quadrilateral be circumscribed about a circle, prove that the middle points of its diagonals and the centre of the circle are in a straight line.

67. From the vertex of the right angle C of a right triangle ACB straight lines CD and CE are drawn, making the angles ACD, ACE each equal to the angle B, and meeting the hypotenuse in D and E. Prove that

$$\overline{DC}^2 : \overline{DB}^2 = AE : EB.$$

68. $ABCD$ is a parallelogram; the circle through A, B, and C cuts AD in A', and DC in C'. Prove that

$$A'D : A'C' = A'C : A'B.$$

69. If two intersecting chords are drawn in a semicircle from the extremities of the diameter, the sum of the products of the segment adjacent to the diameter in each by the whole chord is equal to the square of the diameter.

70. If a quadrilateral circumscribe a circle the two diagonals and the two lines joining the points where the opposite sides of the quadrilateral touch the circle will all four meet in a point.

71. There are two points whose distances from three fixed points are in the ratios $p:q:r$. Prove that the straight line joining them passes through a fixed point whatever be the values of the ratios.

72. The lines joining the vertices of an equilateral triangle ABC to any point D meet the circumscribing circle in the points A', B', C'. Prove that $AD.\,AA'+BD.\,BB'+CD.\,CC'=2\overline{AB}^2$.

73. If from any point perpendiculars are drawn to all the sides of a polygon, the two sums of the squares of the alternate segments of the sides are equal.

74. One circle touches another internally, and a third circle whose radius is a mean proportional between their radii passes through the point of contact. Prove that the other intersections of the third circle with the first two are in a line parallel to the common tangent of the first two.

75. If two tangents are drawn to a circle at the extremities of a diameter, the portion of any third tangent intercepted between them is divided at its point of contact into segments whose product is equal to the square of the radius.

76. A straight line AB is divided harmonically at P and Q; M, N are the middle points of AB and PQ. If X be any point on the line, prove that $XA.\,XB+XP.\,XQ=2XM.\,XN$.

77. The radius of a circle drawn through the centres of the inscribed and any two escribed circles of a triangle is double the radius of the circumscribed circle of the triangle.

78. The centres of the four escribed circles of a quadrilateral lie on the circumference of a circle.

79. O, O_1, O_2, O_3 are the centres of the inscribed and three escribed circles of a triangle ABC. Prove that

$$AO.\,AO_1.\,AO_2.\,AO_3=\overline{AB}^2.\,\overline{AC}^2.$$

80. The opposite sides of a quadrilateral inscribed in a circle, when produced, meet at P and Q; prove that the square of PQ is equal to the sum of the squares of the tangents from P and Q to the circle.

LOCI

81. A is a point on the circumference of a given circle, P a point

without the circle. AP cuts the circle again in B, and the ratio $AP : AB$ is constant.

Find the locus of P.

82. Find the locus of a point whose distances from two given points are in a given ratio.

83. Find the locus of a point the sum of the squares of whose distances from the vertices of a given triangle is constant.

PROBLEMS OF CONSTRUCTION

84. Draw a circle such that, if straight lines be drawn from any point in its circumference to two given points, these lines shall have a given ratio.

85. Construct a triangle, having given the base, the line bisecting the opposite angle, and the diameter of the circumscribed circle.

86. Construct a right triangle, having given the difference between the sides and the difference between the hypotenuse and one side.

87. Construct a triangle, having given the perimeter, the altitude, and that one base angle is twice the other.

88. Construct a triangle, having given an angle, the length of its bisector, and the sum of the including sides.

89. From one extremity of a diameter of a given circle draw a straight line such that the part intercepted between the circumference and the tangent at the other extremity shall be of given length.

90. Divide a semi-circumference into two parts such that the radius shall be a mean proportional between the chords of the parts.

91. Construct a triangle, similar to a given triangle, such that two of its vertices may be on lines given in position, and its third vertex be at a given point.

92. Through four given points draw lines which will form a quadrilateral similar to a given quadrilateral.

93. Find a point such that its distances from three given points may have given ratios.

94. Divide a straight line harmonically in a given ratio.

95. A line perpendicular to the bisector of an angle of a triangle is drawn through the point in which the bisector meets the opposite

side. Prove that the segment on either of the other sides between this line and the vertex is a harmonic mean between those sides.

96. Draw through a given point within a circle a chord which shall be divided at that point in mean and extreme ratio.

PROBLEMS FOR COMPUTATION

97. (1.) The sides of a right triangle are 15 ft. and 18 ft. The hypotenuse of a similar triangle is 20 ft. Find its sides.

(2.) The sides of a right triangle are 16.213 in. and 32.426 in. Find the ratio of the segments of the hypotenuse formed by the altitude upon the hypotenuse.

(3.) In an isosceles triangle the vertex angle is 45°; each of the equal sides is 16 yds. Find the base in meters.

(4.) In a triangle whose sides are 247.93 mm., 641.98 mm., 521.23 mm., find the altitude upon the shortest side.

(5.) In a triangle whose sides are 4, 7, and 9, find the median drawn to the shortest side.

(6.) In a triangle whose sides are 123.41 in., 246.93 in., 157.62 in., compute the bisector of the largest angle.

(7.) Two adjacent sides of a parallelogram are 49 cm. and 53 cm. One diagonal is 58 cm. Find the other diagonal.

(8.) If the chord of an arc is 720 ft., and the chord of its half is 376 ft., what is the diameter of the circle?

(9.) From a point without a circle two tangents are drawn making an angle of 60°. The length of each tangent is 15 in. Find the diameter of the circle.

(10.) Find the radius of a circle circumscribing a triangle whose sides are 35.421 cm., 36.217 cm., 423.92 cm.

BOOK IV

PROBLEMS OF DEMONSTRATION

98. A straight line AB is bisected in C and divided unequally in D. Prove that the sum of the squares on AD and DB is equal to twice the sum of the squares on AC and CD.

99. The area of a triangle is equal to the product of its three sides divided by four times the radius of its circumscribed circle.

100. Prove, by a geometrical construction, that the square on the hypotenuse of a right triangle is equal to four times the triangle plus the square on the difference of the sides.

101. Prove, by a geometrical construction, that the square on the hypotenuse of a right triangle is equal to the square on the sum of the sides minus four times the triangle.

102. On the side BC of the rectangle $ABCD$ as diameter describe a circle. From its centre E draw the radius EG parallel to CD and in the direction C to D. Join G and C by a straight line cutting the diagonal BD in H. From H draw the line HK parallel to CD and in the direction C to D, cutting the circumference of the circle in K. Join BK and produce to meet CD in L. Then CL is the side of a square which is equivalent to the rectangle $ABCD$.

103. Construct any parallelograms $ACDE$ and $BCFG$ on the sides AC and BC of a triangle and exterior to the triangle. Produce ED and GF to meet in H and join HC; through A and B draw AL and BM equal and parallel to HC. Prove that the parallelogram $ALMB$ is equal to the sum of the parallelograms which have been constructed on the sides.

104. If similar triangles be circumscribed about and inscribed in a given triangle, the area of the given triangle is a mean proportional between the areas of the inscribed and circumscribed triangles.

105. Any fourth point P is taken on the circumference of a circle through A, B, and C. Prove that the middle points of PA, PB, PC form a triangle similar to the triangle ABC, of one-fourth the area, and such that its circumscribing circle always touches the given circle at P.

106. Equilateral triangles are constructed on the four sides of a square all lying within the square. Prove that the area of the star-shaped figure formed by joining the vertex of each triangle to the two nearest corners of the square is equal to eight times the area of one of the equilateral triangles minus three times the area of the square.

107. A hexagon has its three pairs of opposite sides parallel. Prove that the two triangles which can be formed by joining alternate vertices are of equal area.

108. A quadrilateral and a triangle are such that two of the sides of the triangle are equal to the two diagonals of the quadrilateral and the angle between these sides is equal to the angle between the diagonals. Prove the areas of the quadrilateral and triangle are equal.

109. Prove that the straight lines drawn from the corners of a square to the middle points of the opposite sides taken in order form a square of one-fifth the area of the original square.

110. The area of the octagon formed by the straight lines joining each vertex of a parallelogram to the middle points of the two opposite sides is one-sixth the area of the parallelogram.

111. $ABCD$ is a parallelogram. A point E is taken on CD such that CE is an n^{th} part of CD; the diagonal AC cuts BE in F. Prove the following continued proportion connecting the areas of the parts of the parallelogram

$$ADEFA : AFB : BFC : CFE = n^2+n-1 : n^2 : n : 1$$

112. The squares $ACKE$ and $BCID$ are constructed on the sides of a right triangle ABC; the lines AD and BE intersect at G; AD cuts CB in H, and BE cuts AC in F. Prove that the quadrilateral $FCHG$ and the triangle ABG are equivalent.

PROBLEMS OF CONSTRUCTION

113. Construct an equilateral triangle which shall be equal in area to a given parallelogram.

114. Construct a square which shall have a given ratio to a given square.

115. A pavement is made of black and white tiles, the black being squares, the white equilateral triangles whose sides are equal to the sides of the squares. Construct the pattern so that the areas of black and white may be in the ratio $\sqrt{3} : 4$.

116. Produce a given straight line so that the square on the whole line shall have a given ratio to the rectangle contained by the given line and its extension. When is the problem impossible?

117. Find a point in the base produced of a triangle such that a straight line drawn through it cutting a given area from the triangle may be divided by the sides of the triangle into segments having a given ratio.

118. Bisect a given quadrilateral by a straight line drawn through a vertex.

PROBLEMS FOR COMPUTATION

119. (1.) If the area of an equilateral triangle is 164.51 sq. in., find its perimeter.

(2.) The perimeter of an equilateral pentagon is 25.135 ft. Its area is 23.624 sq. ft. Find the area of a similar pentagon one of whose sides is 10.361 ft.

(3.) Find, in acres, the area of a triangle, if two of its sides are 16.342 rds. and 23.461 rds., and the included angle is $135°$.

(4.) Find the area of the triangle in the preceding example in hectares.

(5.) The sides of a triangle are 13.461, 16.243, and 20.042 miles. Find the areas of the parts into which it is divided by any median.

(6.) The sides of a triangle are 12 in., 15 in., and 17 in. Find the areas of the parts into which it is divided by the bisector of the smallest angle.

(7.) Two sides of a triangle are in the ratio 2 to 5. Find the ratio of the parts into which the bisector of the included angle divides the triangle.

(8.) The altitude upon the hypotenuse of a right triangle is 98.423 in. One part into which the altitude divides the hypotenuse is four times the other. Find the area of the triangle.

(9.) Find the perimeter of the triangle in the preceding example.

(10.) The areas of two similar polygons are 22.462 sq. in. and 14.391 sq. m. A side of the first is 2 in. Find the homologous side of the second.

(11.) The sides of a triangle are .016256, .013961, and .020202. Find the radius of the inscribed circle.

(12.) A mirror measuring 33 in. by 22 in. is to have a frame of uni-

form width whose area is to equal the area of the mirror; find what the width of the frame should be.

(13.) The sum of the radii of the inscribed, circumscribed, and an escribed circle of an equilateral triangle is unity. What is the area of the triangle?

BOOK V

PROBLEMS OF DEMONSTRATION

120. An equilateral polygon inscribed in a circle is regular. An equilateral polygon circumscribed about a circle is regular, if the number of sides is odd.

121. An equiangular polygon inscribed in a circle is regular if the number of sides is odd. An equiangular polygon circumscribed about a circle is regular.

122. The diagonals of a regular pentagon are equal.

123. The pentagon formed by the diagonals of a regular pentagon is regular.

124. An inscribed regular octagon is equivalent to a rectangle whose sides are equal to the sides of an inscribed and a circumscribed square.

125. If a triangle is formed having as sides the radius of a circle, the side of an inscribed regular pentagon, and the side of an inscribed regular decagon, this triangle will be a right triangle.

126. The area of a regular hexagon inscribed in a circle is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

127. If perpendiculars are drawn from the vertices of a regular polygon to any straight line through its centre, the sum of those which fall upon one side of the line is equal to the sum of those which fall upon the other side.

128. The area of any regular polygon inscribed in a circle is a mean proportional between the areas of the inscribed and circumscribed polygons of half the number of sides.

129. If, on the sides of a right triangle as diameters, semi-circum-

ferences are described exterior to the triangle, and a circumference is drawn through the three vertices, the sum of the crescents thus formed is equivalent to the triangle.

130. If two circles are internally tangent to a third circle and the sum of their radii is equal to the radius of the third circle, the shorter arc of the third circle comprised between their points of contact is equal to the sum of the arcs of the two small circles from their points of contact with the third circle to their intersection which is nearest the large circle.

131. If *CD* is the perpendicular from the vertex of the right angle of a right triangle *ABC*, prove that the areas of the circles inscribed in the triangles *ACD*, *BCD* are proportional to the areas of the triangles.

PROBLEMS OF CONSTRUCTION

132. To construct a circumference whose length shall equal the sum of the lengths of two given circumferences.

133. To construct a circle equivalent to the sum of two given circles.

134. To inscribe a regular octagon in a given square.

135. To inscribe a regular hexagon in a given equilateral triangle.

136. Divide a given circle into any number of parts proportional to given straight lines by circumferences concentric with it.

137. Find four circles whose radii are proportional to given lines, and the sum of whose areas is equal to the area of a given circle.

138. In a given equilateral triangle inscribe three equal circles each tangent to the two others and to two sides of the triangle.

139. In a given circle inscribe three equal circles each tangent to the two others and to the given circle.

140. The length of the circumference of a circle being represented by a given straight line, find approximately by a geometrical construction the radius.

PROBLEMS FOR COMPUTATION

141. (I.) A regular octagon is inscribed in a circle whose radius is 4 ft. Find the segment of the circle contained between one side of the octagon and its subtended arc.

(2.) Find the area of an equilateral triangle circumscribed about a circle whose radius is 14.361 in.

(3.) An isosceles right triangle is circumscribed about a circle whose radius is 3 cm. Find (*a*) each side; (*b*) its area; (*c*) the area in each corner of the triangle bounded by the circumference of the circle and two sides of the triangle.

(4.) Find the area of the circle inscribed in an equilateral triangle, one side of which is 7.4631 ft.

(5.) Find the difference between the area of a triangle whose sides are 4.6213 mm., 3.7962 mm., and 2.6435 mm., and the area of the circumscribed circle.

(6.) The area of a circle is 14632 sq. ft. Find its circumference in yards.

(7.) Find the area of a ring whose outer circumference is 15.437 ft., and whose inner circumference is 9.3421 ft.

(8.) Find the ratio of the areas of two circles inscribed in equilateral triangles, if the perimeter of one triangle is four times that of the other.

(9.) If the area of an equilateral triangle inscribed in a circle is 12 sq. ft., what is the area of a regular hexagon circumscribed about the same circle?

(10.) Find the side of a regular octagon whose area shall equal the sum of the areas of two regular hexagons, one inscribed in and the other circumscribed about a circle whose radius is 10.462 in.

(11.) A man has a circular farm 640 acres in extent. He gives to each of his four sons one of the four largest equal circular farms which can be cut off from the original farm. How much did each son receive?

(12.) A man has a circular tract of land 700 acres in area; he wills one of the three largest equal circular tracts to each of his three sons, the tract at the centre included between the three circular tracts to his daughter, and the tracts included between the circumference of the original tract and the three circular tracts to his wife. How much will each receive?

(13.) A man owned a tract of land 323,250 sq. m. in area, and in the

form of an equilateral triangle. To each of his three sons he gave one of the three largest equal circular tracts which could be formed from the given tract; to each of his three daughters one of the corner sections cut off by a circular tract; to each of his three grandchildren one of the side sections cut off by two of the circular tracts; he himself retained the central section included between the three circular tracts. Find the share of each.

BOOK VI

PROBLEMS OF DEMONSTRATION

142. If a straight line is parallel to a plane, the shortest distance of the line from all straight lines of the plane which are not parallel to it is the same.

143. If a straight line is parallel to a plane it is everywhere equidistant from the plane.

144. If a plane is passed through two vertices of a parallelogram, the perpendiculars to it from the other vertices are equal.

145. If from the foot of a perpendicular to a plane a straight line is drawn perpendicular to any line of the plane, and the intersection of these lines is joined to any point of the perpendicular to the plane, the last line will be perpendicular to the line of the plane.

146. The plane angle of a right diedral angle is a right angle, and conversely. Two diedral angles are to each other as their plane angles.

147. If a line is drawn in each face plane of any triedral angle through its vertex and perpendicular to the opposite edge, prove that these three lines lie in the same plane.

148. A, B, C are points on the three edges of a triedral angle of which the face angles are right angles; S is the projection of the vertex O on the plane ABC. Prove that the triangle AOB is a mean proportional between the triangles ABC and ASB.

LOCI

149. Find the locus of a point in space the difference of the squares of whose distances from two given points is constant.

150. *Defs.*—The **angle** between two straight lines not in the same plane, that is, neither parallel nor intersecting, is the angle between two lines drawn through any point in space parallel respectively to the two lines and in the same directions.

Two straight lines in space are **perpendicular** when their angle, as defined above, is a right angle.

151. Find the locus of the middle point of a straight line of given length which has its extremities upon two given perpendicular but non-intersecting straight lines.

152. A straight line moves parallel to a fixed plane and intersects two fixed straight lines not in the same plane. Find the locus of a point which divides the part intercepted in a constant ratio.

153. Find the locus of a point in a given plane such that the straight lines joining it to two given points not in the plane make equal angles with the plane.

154. Find the locus of a point the sum of whose distances from two given planes is equal to a given straight line.

155. Find the locus of a point equidistant from the three faces of a triedral angle.

PROBLEMS OF CONSTRUCTION

156. Draw a line in a given plane, and through a given point in the plane, which shall be perpendicular to a given straight line in space.

157. Pass a plane cutting the faces of a polyedral angle of four faces in such a manner that the section shall be a parallelogram.

158. Given a straight line AB parallel to a plane M. From any point A in AB draw a straight line AX, of given length, to the plane M, so as to make the angle BAX equal to a given angle.

159. Through a given point in a plane, to draw a straight line in that plane which shall be at a given distance from a given point outside of the plane.

160. A given straight line intersects a given plane. Through the intersection draw a straight line in the given plane, making a given angle with the given line.

161. Given three straight lines in space. Draw a straight line from the first to the second parallel to the third.

162. Given two straight lines not in the same plane. Find a point in one at a given perpendicular distance from the other.

163. Through a given point draw a straight line to meet a given straight line and the circumference of a given circle not in the same plane with the given line.

BOOK VII

PROBLEMS OF DEMONSTRATION

164. A triangular pyramid is cut by a plane parallel to the base, and a plane is passed through each vertex of the base and the points where the cutting plane meets the two opposite lateral edges. Determine the locus of the point of intersection of the three planes thus passed.

165. At any point in the base of a regular pyramid a perpendicular to the base is erected, intersecting the lateral faces of the pyramid, or these faces produced. Prove that the sum of the perpendicular distances from the points of intersection to the base is constant.

166. The perpendicular from the centre of gravity of a tetraedron (§ 749) to any plane is one-fourth the sum of the four perpendiculars from the vertices of the tetraedron to the same plane.

167. If the edges of a hexaedron meet four by four in three points, the four diagonals of the hexaedron meet in a point.

168. Prove that straight lines through the middle points of the sides of any face of a tetraedron each parallel to the straight line connecting a fixed point D with the middle point of the opposite edge, meet in a point E such that DE passes through and is bisected by the centre of gravity of the tetraedron.

169. The sum of the perpendiculars drawn to the faces of a regular tetraedron from any point within is equal to the altitude of the tetraedron.

170. A regular octaedron is cut by a plane parallel to one of its faces; prove that the perimeter of the section is constant.

171. In a tetraedron the sum of two opposite edges is equal to the sum of two other opposite edges. Prove that the sum of the diedral angles whose edges are the first pair of lines is equal to the sum of the diedral angles whose edges are the other pair of lines.

172. C' and D' are the feet of the perpendiculars drawn from any point to the faces opposite the vertices C, D of a tetraedron $ABCD$.

Prove that $\overline{AC'}^2-\overline{BC'}^2=\overline{AD'}^2-\overline{BD'}^2$.

173. If the opposite edges of a tetraedron are perpendicular to each other, the perpendiculars drawn from the vertices to the opposite faces meet in a point.

174. If a tetraedron is cut by a plane which passes through the middle points of two opposite edges, the section is divided into two equivalent triangles by the straight line joining these points.

175. From the middle point of one of the edges of a regular tetraedron a fly descends by crawling around the tetraedron, and reaches the base at the point where this edge meets the base. Find at what points the fly must cross the other edges if its path is everywhere equally inclined to the plane of the base.

176. The plane bisecting a diedral angle of a tetraedron divides the opposite edge into segments which are proportional to the faces which include the diedral angle.

177. Straight lines are drawn from the vertices A, B, C, D of a tetraedron through a point P, to meet the opposite faces in A', B', C', D'.

Prove that $$\frac{PA'}{AA'}+\frac{PB'}{BB'}+\frac{PC'}{CC'}+\frac{PD'}{DD'}=1.$$

178. If a is the edge of a regular tetraedron, its volume is $\frac{a^3}{12}\sqrt{2}$.

179. If a is the edge of a regular octaedron, its volume is $\frac{a^3}{3}\sqrt{2}$.

180. The lateral surface of a pyramid is greater than its base.

181. The volume of a triangular prism is equal to the area of a lateral face multiplied by one-half its perpendicular distance from any point in the opposite lateral edge.

182. The volume of a regular prism is equal to the product of its lateral area by one-half the apothem of its base.

183. The three lateral faces of a tetraedron are perpendicular to each other. If a triangle drawn in the base is projected on each of the three lateral faces, prove that the sum of the pyramids having these projections as bases and a common vertex anywhere in the base of the given tetraedron is equivalent to the pyramid having the given triangle for its base and its vertex at the vertex of the given tetraedron.

184. Extend the last exercise to the case where the common vertex is at any point in the plane of the base by regarding the volume of a pyramid as negative if the altitude is in the opposite direction from that in which it was measured for the pyramid on the same base in the last exercise.

185. *Defs.*—If $ABCD$ is a rectangle, and EF a straight line parallel to AB, and not in the plane of the rectangle, the solid bounded by the rectangle $ABCD$, the trapezoids $ABFE$, $CDEF$, and the triangles ADE. BCF is a **wedge.**

The rectangle is called the **back** of the wedge; the trapezoids, its **faces**; the triangles, its **ends**; the line EF, its **edge**; AB is the length of the back and AD its breadth; the perpendicular from any point of EF upon the back is the **altitude** of the wedge.

186. If h is the altitude, prove that the volume of the above wedge is

$$\tfrac{1}{6}h \times AD \times (2AB + EF).$$

187. *Defs.*—If $ABCD$ and $EFGH$ are two rectangles lying in parallel planes, AB and BC being parallel to EF and FG, respectively, the solid bounded by these two rectangles and the trapezoids $ABFE$, $BCGF$, $CDHG$, $DAEH$, is called a **rectangular prismoid.** The rectangles are called the **bases** of the prismoid and the perpendicular distance between them the **altitude.**

188. Prove that the volume of a rectangular prismoid is equal to the product of the sum of its bases, plus four times a section equidistant from the bases, multiplied by one-sixth the altitude.

PROBLEMS OF CONSTRUCTION

189. Having given the four perpendiculars from the vertices of a tetraedron to the opposite faces, and the distance of a point in space from three of the faces, find its distance from the fourth face.

190. Through a given straight line in one of the faces of a tetraedron pass a plane which shall cut off from the tetraedron another tetraedron which is to the first in a given ratio.

191. Find two straight lines whose ratio shall be the ratio of the volumes of two given cubes.

192. Find a point within a given tetraedron, such that the four pyramids having this point for vertex, and the faces of the tetraedron for bases, shall be equivalent.

PROBLEMS FOR COMPUTATION

193. (1.) Find the lateral area, total area, and volume of a regular triangular prism the perimeter of whose base is 16.413 in. and whose altitude is 14.718 in.

(2.) Find the lateral area, total area, and volume of a regular hexagonal pyramid each side of whose base is 8.84 in. and whose altitude is 4.92 in.

(3.) The area of the base of a pyramid is 13 sq. m.; its altitude is 4 m. Find the area of a section parallel to the base and distant $1\frac{1}{2}$ m. from it. Also find the volume of the pyramid cut off by this plane.

(4.) Find the volume of a frustum of a pyramid whose base is a regular octagon having each side equal to 4 in., and whose altitude is 9 in., made by a plane 5 in. from the vertex.

(5.) The diagonal of a cube is 24.16 cm. Find its surface and volume.

(6.) The volume of a polyedron is 984.62 cu. ft. Find the volume of a similar polyedron whose edges are nine times the edges of the first polyedron.

(7.) The volume of a given tetraedron is 6.86 cu. m. Find the volume of the tetraedron whose vertices are a vertex of the given tetraedron and the intersections of the medians of the faces including that vertex.

(8.) Find the surface and volume of a regular tetraedron whose edge is 1.

(9.) Find the surface and volume of a regular octaedron whose edge is 16.247 mm.

(10.) Find the ratio of the volumes of a cube and a regular tetraedron whose edges are equal.

(11.) Find the ratio of the volumes of a regular octaedron and a regular tetraedron whose edges are equal.

(12.) Find the number of cubic feet of water that will be contained by a trench in the shape of a wedge the length of whose back is 20 m., whose breadth is 3 m., whose edge is 16 m., and whose depth is $2\frac{1}{2}$ m. How many pounds of water will the trench hold, each cubic foot of water weighing $62\frac{1}{2}$ lbs.? How many metric tons?

(13.) An embankment is in the form of a rectangular prismoid. The length and breadth of its base are 246 ft. and 8 ft.; the length and breadth of its top are 239 ft. and 3 ft. Its height is 4 ft. Find the number of cubic yards of earth it contains.

BOOK VIII

PROBLEMS OF DEMONSTRATION

194. If two circles in space are such that their centres are the projections of the same point on their planes, and the tangents to the circles drawn from a point in the intersection of their planes are equal, the two circles are on the same sphere.

195. If through a fixed point within or without a sphere three straight lines are drawn at right angles to each other so as to intersect the surface of the sphere, the sum of the squares of the three chords thus formed is constant. Also the sum of the squares of the six segments of these chords is constant.

196. If three radii of any sphere perpendicular to each other are projected upon any plane, the sum of the squares of the three projections is equal to twice the square of the radius of the sphere.

197. If from a point without a sphere any number of straight lines be drawn to touch the sphere, the points of contact will all be in one plane.

198. A sphere can be inscribed in or circumscribed about any regular polyedron.

199. A regular tetraedron and a regular octaedron are inscribed in the same sphere; compare the radii of the spheres which can be inscribed in the tetraedron and in the octaedron.

200. From any point P in a diameter of a given sphere straight lines PQ, PR are drawn perpendicular to that diameter, in any direction and of any length, provided Q and R lie within the sphere. Through P, Q, R two spherical surfaces are passed touching the given spherical surface. Prove that the sum of their radii is equal to the radius of the given sphere.

201. If a square is inscribed in a face of a cube, the plane determined by one side of the square and the corner of the opposite face in the same edge as the adjacent corner of the same face, touches the inscribed sphere.

202. If the opposite edges of a tetraedron are equal, its four vertices may be taken as four non-adjacent vertices of a rectangular parallelopiped. Prove that of the five spheres touching the faces of such a tetraedron, or the faces produced, four have their centres at the remaining four vertices of the parallelopiped, and the fifth at the intersection of the diagonals of the parallelopiped.

203. ABC is a spherical triangle and AT an arc of a great circle tangent to the circumscribing small circle at A. Prove that the angle BAT is equal to the angle C minus half the spherical excess of the triangle.

204. ABC is a spherical triangle; through the middle points of AB and AC an arc of a great circle is drawn cutting BC produced in D. Prove that DB is the supplement of DC.

205. How many spheres, each equal to a given sphere, can be tangent to the given sphere at the same time?

LOCI

206. Find the locus of a point whose distances from three given points in space are in the ratio of three given lines.

207. Find the locus of the intersection of planes tangent to a sphere at the extremities of chords which pass through a fixed point.

208. Find the locus of a point which divides in a given ratio a straight line drawn from a fixed point to the surface of a sphere.

PROBLEMS OF CONSTRUCTION

209. Find the centre of a sphere which passes through the circumference of a given circle, and through a given point not in the plane of the circle.

210. Through a given point pass two spherical surfaces tangent to a given sphere.

211. Find the radius of a sphere which shall circumscribe four equal spheres which touch each other.

PROBLEMS FOR COMPUTATION

212. (1.) The area of the base of a circular cone is 43 sq. in. Its altitude is 19 in. Find the area of a section parallel to the base and 10 in. from the vertex.

(2.) If the area of a circle of a sphere distant 10 cm. from its centre is 40 sq. cm., find the radius of the sphere.

(3.) The polar distance of a circle of a sphere is 30°. If its circumference is 6 m., what is the radius of the sphere?

(4.) Find the area in square feet of a lune whose angle is 36° on a sphere whose surface is 46 sq. m.

(5.) If the area of a spherical triangle whose angles are 110°, 46°, and 150°, is 84.662 sq. yds., what is the area of a trirectangular triangle on the same sphere?

(6.) The angles of a spherical pentagon are 68°, 97°, 156°, 80°, and 142°. Its area is 8 sq. ft. Find the area of the sphere.

(7.) Find the volume of a spherical ungula the angles of whose base are each 43°, in a sphere whose volume is 18.561 cu. m.

(8.) Find the volume of a spherical pyramid the angles of whose base are 70°, 98°, 153°, 89°, and 150°, in a sphere whose volume is 77.253 cu. yds.

(9.) The radii of two spheres are 14 m. and 9 m. respectively. The distance between their centres is 20 m. Find the length of the circumference in which their surfaces intersect.

(10.) Find the radii of the spheres inscribed in and circumscribed about a regular tetraedron whose edge is 6.5438 in.

(11.) If the radius of the sphere circumscribed about a cube is 10.643 ft., find the volume of the cube.

(12.) Find the surface of a regular octaedron, the radius of whose circumscribed sphere is 32.147 in.

(13.) A hollow cone of revolution of which the altitude is equal to three-fourths the slant height is cut open in a straight line drawn from the vertex to a point in the base. Find (in right angles) the vertical angle of the unrolled surface.

BOOK IX

PROBLEMS OF DEMONSTRATION

213. The volume of a cylinder of revolution is equal to its lateral area multiplied by one-half the radius of its base.

214. The volume of a cylinder of revolution is equal to the area of its generating rectangle multiplied by the circumference of a circle whose radius is the distance from the centre of the rectangle to the axis.

215. The volume of a cone of revolution is equal to the area of its generating triangle multiplied by the circumference of a circle whose radius is the distance from the intersection of the medians of the triangle to the axis.

216. Express the volume of a cone of revolution in terms of its lateral area and the perpendicular from the centre of its base upon an element.

217. Express the volume of a cone of revolution in terms of its total surface and the radius of the inscribed sphere.

218. The volumes of polyedrons circumscribed about equal spheres are proportional to their surfaces.

219. Two spheres intersect, the centre of the first lying on the surface of the second. Prove that the surface intercepted by the first on the second is independent of the size of the second sphere. Prove that this surface is one-fourth the surface of the first sphere.

PROBLEMS OF CONSTRUCTION

220. Cut a sphere by a plane so that the area of the section shall be equal to the difference of the areas of the two zones which the plane determines.

221. Divide a zone in mean and extreme ratio by a plane parallel to its bases.

222. Inscribe in and also circumscribe about a sphere a cone of which the total area shall be in a given ratio to the area of the sphere.

223. Inscribe in and also circumscribe about a given sphere a cone of which the volume shall be in a given ratio to the volume of the sphere.

224. Determine a point on the diameter of a sphere such that if a plane is passed through this point perpendicular to the diameter, the surface of the zone limited by this plane and containing the nearer extremity of the diameter shall be equal to the lateral surface of the cone whose base is the circle of intersection of the plane with the sphere and whose vertex is the farther extremity of the diameter.

PROBLEMS FOR COMPUTATION

225. (1.) If the perimeter of a right section of a cylinder is 16 in., and its lateral area is 256 sq. in., what is the length of an element?

(2.) Find the volume of a cylinder of revolution whose total area is 160π and whose radius is 4.

(3.) Find the radius of a cylinder of revolution whose total area is 80π and whose altitude is 6.

(4.) An oil tank is in the form of a circular cylinder. If the tank is 26 ft. long and 78 in. in diameter, how many liters of oil will it contain?

(5.) Find the volume of a cone of revolution whose total area is 200π and whose altitude is 16.

(6.) The lateral area of a cone of revolution is 39π. Its altitude is 9. Find the height of an equivalent cylinder of revolution whose radius is 4.

(7.) In a sphere whose diameter is 14 in. the altitude of a zone of

one base is 2 in. Find the altitude of a cylinder of revolution whose lateral area shall equal the area of the zone and whose base shall equal the base of the zone.

(8.) Find the radius of a circle whose area shall equal the area of a zone of altitude 16.954 m. on a sphere whose diameter is 20 m.

(9.) Find the radius of a sphere whose area shall equal the area of the zone in the previous example.

(10.) A conical glass is 5 in. high and 4 in. across at the top. A marble is within the glass and water is poured in till the marble is just immersed. If the amount of water poured in is $\frac{1}{4}$ the contents of the glass, what is the diameter of the marble?

(11.) If two spheres of radii 13 in. and 8 in. are inscribed in a cone of revolution so that the greater may touch the less and also the base of the cone, find the volume of the cone.

(12.) A sphere and an octaedron are inscribed in the same cube, the vertices of the octaedron being at the centres of the faces of the cube. Compare the volumes of the three solids.

(13.) Find the ratio of the volume of a sphere touching the edges of a regular tetraedron to the volume of a sphere touching one face and the other faces produced.

(14.) The volume of a spherical sector is 19.463 cu. mm. Its base is one-third the surface of the sphere. Find the surface of the sphere.

(15.) Find the volume of a spherical shell whose two surfaces are 20π and 15π.

(16.) Find the volume of a spherical segment whose altitude is 9 in. and the radii of whose bases are 4 in. and 5 in.

(17.) Assuming the diameter of the earth to be 7960 miles, what is the area of the portion which would be visible from a point 3980 miles above its surface?

(18.) Show that if R is the radius of the earth and h the height of a point of observation above its surface, the area of the visible surface is $\dfrac{2\pi R^2 h}{R+h}$.

(19.) In a sphere whose radius is 20 in. find the volume of a segment of one base whose altitude is 6 in.

(20.) Show that if R is the radius of a sphere and h the altitude of a spherical segment of one base, the volume of the spherical segment is $\pi h^2(R-\frac{1}{3}h)$.

PROBLEMS IN MAXIMA AND MINIMA IN PLANE AND SOLID GEOMETRY

226. Through a given point draw a straight line which shall form with two given lines a triangle of minimum area.

227. Through a given point within a given angle draw a straight line which shall form with the sides of the given angle a triangle of minimum perimeter.

228. Through the intersection of two tangents to a circle draw a straight line cutting the circumference in two points such that, if they are joined to the points of tangency, the product of either pair of opposite sides of the inscribed quadrilateral thus formed shall be a maximum.

229. In an acute-angled triangle inscribe a rectangle, such that its diagonal shall be a minimum.

230. From a given point in a diameter AB of a circle produced draw a straight line cutting the circumference in two points C and H, so that the triangle ACH shall be a maximum.

231. Two straight lines containing a given angle are drawn from a given point in the base of a triangle, forming a quadrilateral with the two other sides. Prove that, of all the quadrilaterals which may be thus formed, that one whose sides passing through the given point are equal is a maximum, if the given angle is less than the supplement of the opposite angle of the triangle; a minimum, if the given angle is greater than the supplement of the opposite angle; neither a maximum nor a minimum, if the given angle is equal to the supplement of the opposite angle.

232. In the last exercise a maximum or a minimum quadrilateral can be formed for each point in the base (except in the case when the given angle is the supplement of the opposite angle of the triangle). Prove that, of all these maximum or minimum quadrilaterals, the least maximum or the greatest minimum is that whose equal sides make equal angles with the base.

233. Find a point in a plane such that the sum of its distances from two fixed points on the same side of the plane shall be a minimum.

234. Find a point in a plane such that the difference of its distances from two fixed points on opposite sides of the plane shall be a maximum.

235. Of all quadrangular prisms of which the volumes are equal, the cube has the least surface.

INTRODUCTION TO

MODERN GEOMETRY

[The numbers of the figures are the same as of the articles to which they belong.]

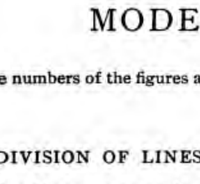
DIVISION OF LINES. THE COMPLETE QUADRILATERAL

1. The lines connecting any point with the three vertices of a triangle so divide the opposite sides that the product of three non-adjacent segments is equal to the product of the other three non-adjacent segments.

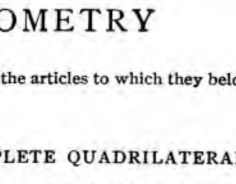
Hint.—Draw MCN parallel to AB.

From similar triangles,

$$\frac{AZ}{ZB}\cdot\frac{BX}{XC}\cdot\frac{CY}{YA}=\frac{CM}{CN}\cdot\frac{AB}{CM}\cdot\frac{CN}{AB}=1.$$



FIGS. 1 AND 2



FIGS. 1 AND 2

2. Conversely, if the sides of a triangle are so divided (either two or not any of the points of division being on the sides produced) that the product of three non-adjacent segments is equal to the product of the other three non-adjacent segments, the lines connecting the points of division with the opposite vertices meet in a point.

Hint.—Use the method of reductio ad absurdum.

3. *Def.*—A line which cuts a system of lines is a **transversal.**

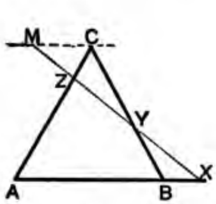
In § 4 and § 5 XZ is a transversal which cuts the lines AB, AC, BC.

4. If the sides of a triangle are cut by a transversal, the product of three non-adjacent segments is equal to the product of the other three non-adjacent segments.

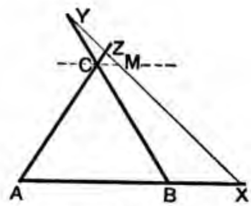
Hint.—Draw CM parallel to AB.

From similar triangles,

$$\frac{AX}{XB}\cdot\frac{BY}{YC}\cdot\frac{CZ}{ZA}=\frac{AX}{XB}\cdot\frac{XB}{CM}\cdot\frac{CM}{AX}=1.$$



FIGS. 4 AND 5



FIGS. 4 AND 5

5. Conversely, if the sides of a triangle are so divided (either one or three of the points of division being on the sides produced) that the product of three non-adjacent segments is equal to the product of the other three non-adjacent segments, the points of division are in a straight line.

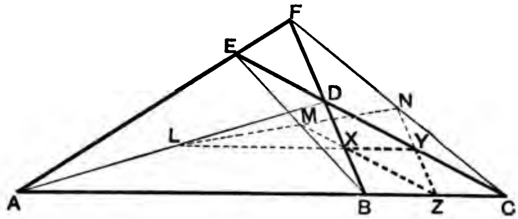
Hint.—Use the method of reductio ad absurdum.

6. *Exercise.*—If $ABCD$ be four points taken in order on a straight line,
$$AB.CD+BC.AD=AC.BD.$$

FIG. 6

7. *Def.*—A **complete quadrilateral** is the figure formed by four straight lines intersecting in six points. The six points are the vertices; the three lines connecting opposite vertices are the diagonals.

$ABCDEF$ is a complete quadrilateral; AD, BE, CF, are the diagonals.



FIGS. 7 AND 8

8. The middle points of the diagonals of a complete quadrilateral are in a straight line.

Hint.—Let L, M, N be the middle points of the diagonals. Construct the triangle XYZ, whose vertices are at the middle points of BD, DC, CB; the sides of this triangle pass through L, M, N.

Since FA is a transversal cutting the sides of the triangle BCD,

$$DF.BA.CE = FB.AC.ED. \qquad \S\ 4$$

But $\quad YN = \frac{1}{2}DF,\ NZ = \frac{1}{2}FB$, etc.

Hence $\quad YN.XL.ZM = NZ.LY.MX.$

Therefore the points L, M, N, being on the sides of the triangle XYZ, are in a straight line. § 5

HARMONIC SECTION

9. *Def.*—If a line AB is divided harmonically at C and D, the points C and D are **harmonic conjugates** to the points A and B. The four points A, B, C, D are **harmonic points**, and AB is a **harmonic mean** between AC and AD.

A line is divided harmonically if it is divided internally and externally in the same ratio. § 332, p. 151

Thus, if $$\frac{AC}{CB} = \frac{AD}{DB},$$

AB is divided harmonically at C and D.

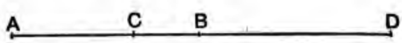


FIG. 9

10. *Exercise.*—The above definition of a harmonic mean is equivalent to the algebraic definition.

Hint.—In Algebra the harmonic mean between a and b is $\frac{2ab}{a+b}$.

11. *Def.*—A **pencil of rays** is a system of straight lines (rays) passing through a point (the vertex).

Thus OA, OB, OC, OD, Fig. 13, form a pencil of rays.

12. *Def.*—A **harmonic pencil** is a pencil of four rays which pass through the harmonic points of a line.

13. Any transversal is cut harmonically by a harmonic pencil.

Hint.—Let A, B, C, D be harmonic points, P and p the perpendiculars from O on AD and ad. To prove $\frac{ac}{cb}=\frac{ad}{db}$, that is, $\frac{ac.db}{ad.cb}=1$.

The ratio of the areas of two corresponding triangles as aOc and AOC is

$$\frac{Oa.Oc}{OA.OC}=\frac{\frac{1}{2}p.ac}{\frac{1}{2}P.AC}. \qquad \text{§ 398, p. 180}$$

$$\text{Hence} \quad \frac{ac.db}{ad.cb}=\frac{\dfrac{Oa.Oc.Od.Ob}{OA.OC.OD.OB}}{\dfrac{Oa.Od.Oc.Ob}{OA.OD.OC.OB}}\times\frac{AC.DB}{AD.CB}=\frac{AC.DB}{AD.CB}=1.$$

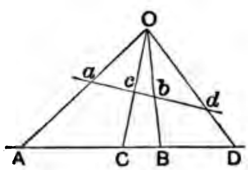


FIG. 13

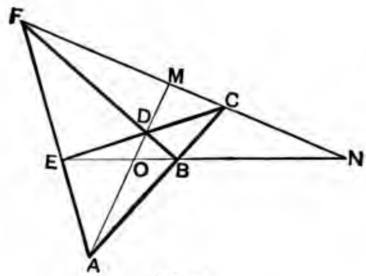


FIG. 14

14. Each diagonal of a complete quadrilateral is divided harmonically by the other two.

Hint.—Since BN is a transversal cutting the sides of the triangle ACF,

(1) $AB.CN.FE=BC.NF.EA.$ § 4

(2) Also $AB.CM.FE=BC.MF.EA.$ § 1

By dividing (1) by (2) $\frac{CN}{CM}=\frac{NF}{MF}.$

Therefore $\frac{CM}{MF}=\frac{CN}{NF}.$

15. **If two harmonic pencils have one pair of corresponding rays coincident, the intersections of the other three pairs of corresponding rays are in a straight line.**

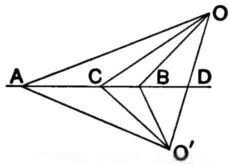


FIG. 15

Hint.—Use the method of reductio ad absurdum.

SOME PROPERTIES OF CIRCLES

16. **The product of the perpendiculars drawn from a point on a circle* to two tangents is equal to the square of the perpendicular drawn from the point to their chord of contact.**

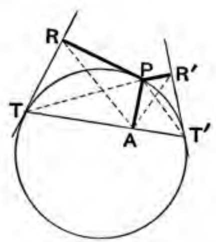


FIG. 16

Hint.—Let RT, $R'T'$ be the tangents and TT' their chord of contact.

A circle can be circumscribed about each of the quadrilaterals $APRT$ and $APR'T'$, since the sum of the opposite angles in each is equal to two right angles.

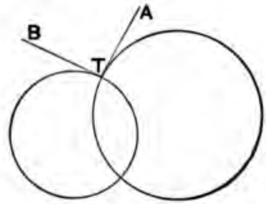
Hence angle $ARP = ATP = PT'R' = R'AP$. Likewise angle $AR'P = PAR$.

Therefore the triangles ARP and $R'AP$ are similar and $AP^2 = PR.PR'$.

* The word circle instead of circumference is used except where ambiguity would result.

17

17. *Def.*—The angle at which two circles cut each other is the angle between the tangents drawn at the point of intersection.



FIGS. 17 AND 18

18. *Def.*—If two circles cut each other at right angles they are said to cut **orthogonally**.

19. If the square of the distance between the centres of two circles is equal to the sum of the squares of their radii, the circles cut each other orthogonally and conversely.

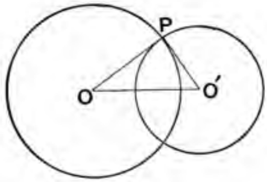


FIG. 19

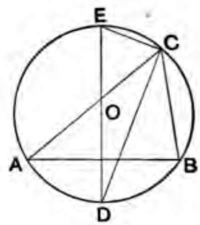


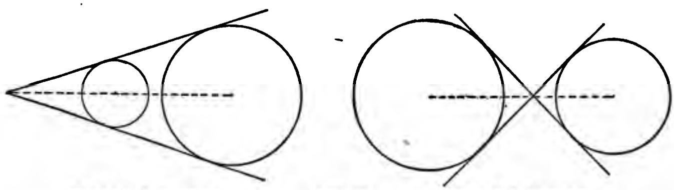
FIG. 20

20. If a circle be circumscribed about a triangle, the lines joining the extremities of the diameter which is perpendicular to the base, to the vertex, are the internal and external bisectors of the vertex angle.

Hint.—Angle DCE is a right angle.

21. *Defs.*—The point of intersection of the direct common tangents of two circles is their **external centre of similitude**; the point of intersection of their inverse common tangents, their **internal centre** of **similitude.**

The centres of similitude are on the line of centres of the circles, and divide that line externally and internally in the ratio of the radii.



EXTERNAL CENTRE FIG. 21 INTERNAL CENTRE

22. The six centres of similitude of three circles are three by three on four straight lines.

The three external centres of similitude are in a straight line, and each pair of internal centres of similitude is in a straight line passing through an external centre of similitude.

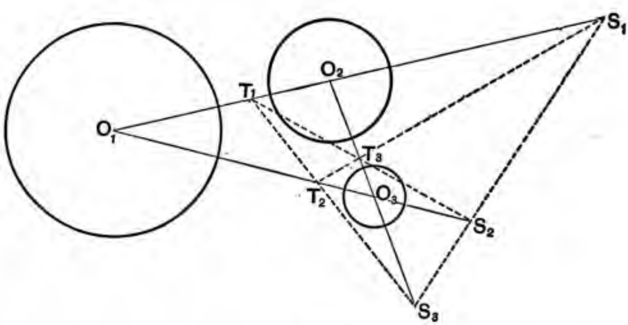


FIG. 22

Hint.—Let S_1, S_2, S_3 be the external, T_1, T_2, T_3 the internal centres of similitude; also let R_1, R_2, R_3 be the radii of the circles.

$$\frac{S_1O_1}{S_1O_2} \cdot \frac{S_2O_3}{S_2O_1} \cdot \frac{S_3O_2}{S_3O_3} = \frac{R_1}{R_2} \cdot \frac{R_3}{R_1} \cdot \frac{R_2}{R_3} = 1.$$

Hence S_1, S_2, S_3 are in a straight line. § 5

Again $$\frac{S_3O_2}{S_3O_3} \cdot \frac{T_1O_1}{T_1O_2} \cdot \frac{T_2O_3}{T_2O_1} = \frac{R_2}{R_3} \cdot \frac{R_1}{R_2} \cdot \frac{R_3}{R_1} = 1.$$

Hence T_1, T_2, S_3 are in a straight line. § 5

23. *Cor.*—If a variable circle touch two fixed circles, the chord of contact passes through an external centre of similitude of the fixed circles; for each point of contact is a centre of similitude of the variable circle and one of the fixed circles.

INVERSION

24. *Def.*—Two points are inverse to each other with respect to a given centre of inversion if they are in the same straight line with this centre, and if the product of their distances from it is equal to a constant.

Two curves are inverse to each other if the successive points of the one invert into the successive points of the other with respect to a given centre.

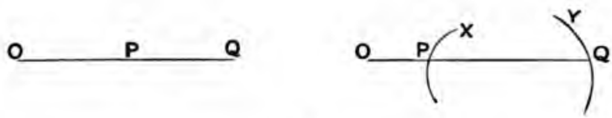


FIG. 24

Q is the inverse of P with respect to the centre, O if $OP=\dfrac{K}{OQ}$.

The curve Y is the inverse of the curve X, if, for every point P of X there is a point Q of Y such that $OP.OQ=K$.

25. The inverse of a circle is a straight line if the centre of inversion is on the circle.

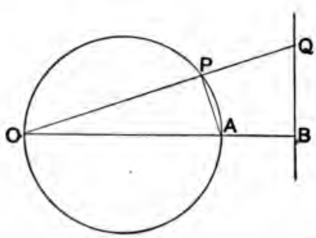


FIG. 25

Hint.—$OP.OQ=OA.OB=K$.

26. This principle makes it possible to draw a line mathematically straight; for in the four linkages* shown below the point P inverts into Q with respect to the centre O, and if P move in a circle passing through the fixed point O, then Q will move in a straight line.

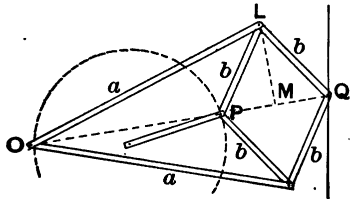


FIG 26 (1)

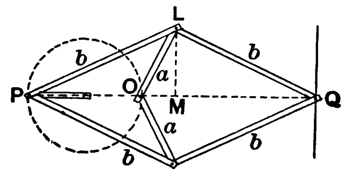


FIG. 26 (2)

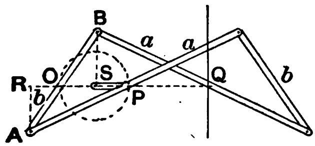


FIG. 26 (3)

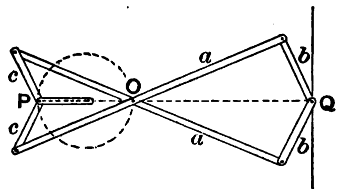


FIG. 26 (4)

In each linkage the bars (links) denoted by the same letter are equal.
To prove the property of inversion:

In linkages (1) and (2)

$QP = OM - PM$, and $OQ = OM + PM$; then $OP.OQ = OM^2 - PM^2$.

But $\quad OM^2 = a^2 - LM^2$ and $PM^2 = b^2 - LM^2$.

Therefore $\quad OP.OQ = a^2 - b^2$, a constant.

In linkage (3) the points O, P, Q divide the links in the same ratio.

$$OP = RP - RO = \sqrt{AP^2 - AR^2} - \sqrt{AO^2 - AR^2},$$

and $$OQ = OS + SQ = \sqrt{BO^2 - BS^2} + \sqrt{BQ^2 - BS^2}.$$

But $$BS = \frac{BO}{AO} \cdot AR, \quad BQ = \frac{BO}{AO} \cdot AP.$$

Therefore $$OP.OQ = \frac{BO}{AO}\left\{AP^2 - AO^2\right\}, \text{ a constant.}$$

Compare linkage (4) with the pantograph. p. 139

* A linkage is a system of bars pivoted together.

The original account of linkages (1) and (2) was published in "Nouvelles Annales," 1873; of (3) in the "Report of the British Association," 1874; of (4) in the "Report of the British Association," 1884.

27. The inverse of a circle is a second circle if the centre of inversion is not on the first circle.

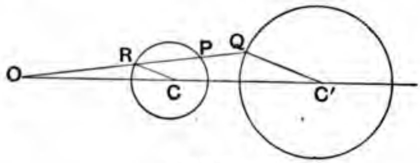


FIG. 27

Hint.—Let C be the first circle, O the centre of inversion, Q the inverse of P. Draw QC' parallel to RC to meet OC in C'.

Since $OP.OR$ is constant and $OP.OQ$ is constant, $\dfrac{OQ}{OR}$ is constant.

By similar triangles $\dfrac{OC'}{OC}=\dfrac{OQ}{OR}$ and $\dfrac{C'Q}{CR}=\dfrac{OQ}{OR}$.

Therefore C' is a fixed point and $C'Q$ a constant length.

28. *Exercise.*—The centre of inversion is a centre of similitude of a given circle and its inverse.

29. *Exercise.*—If two circles touch each other, their inverses also touch each other.

30. *Exercise.*—A circle can be inverted into itself.

Hint.—The constant of inversion must be equal to the square of the tangent drawn to the circle from the centre of inversion.

31. The inverse of a sphere is a plane, if the centre of inversion is on the sphere.

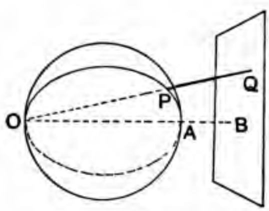


FIG. 31

Hint.—Every point on the sphere is on a great circle passing through the centre of inversion, and will invert into a point of the plane; compare with § 25.

32. The inverse of a sphere is a second sphere, if the centre of inversion is not on the first sphere.

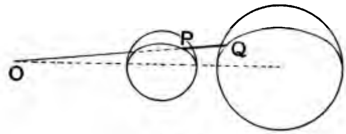


FIG. 32

Hint.—Compare with § 27.

33. If two circles intersect, their angle of intersection is equal to the angle of intersection of their inverses.

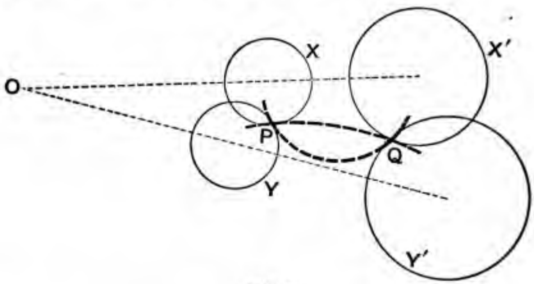


FIG. 33

Hint.—The circles X and Y invert into X' and Y'.

A circle can be described tangent to X at P and passing through Q. This circle inverts into itself and is therefore tangent to X' at Q. §§ 30, 29

Likewise a circle can be described tangent to Y at P and passing through Q. This circle is tangent to Y' at Q.

The angle at which these tangent circles intersect is equal to the angle at which X and Y intersect and also to the angle at which X' and Y' intersect.

34. *Cor.*—If a straight line and a circle, or two straight lines, intersect, their angle of intersection is equal to the angle of intersection of their inverses.

35. A single inversion may be found equivalent to any series of an odd number of inversions from the same centre.

Hint.—If a invert into b, b into c, c into d, . . . m into n where the number of inversions is odd, find an inversion by which a inverts into n.

Why does not this theorem apply to an even number of inversions?

RADICAL AXIS AND COAXAL CIRCLES

36. The locus of a point, from which tangents drawn to two circles are equal, is a straight line perpendicular to the line of centres.

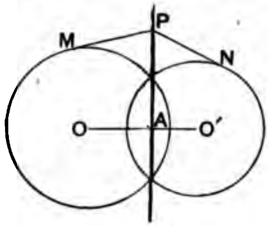


FIG. 36 (1)

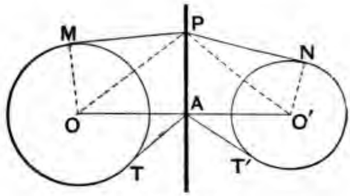


FIG. 36 (2)

Hint.—(1.) If the circles intersect, the locus is the common chord.

(2.) If the circles do not intersect, let A be the point in the line of centres from which tangents to the circles are equal. Erect the perpendicular AP.

$$PM^2 = PO^2 - OM^2$$
$$= AO^2 + AP^2 - OM^2$$
$$= AT^2 + OM^2 + AP^2 - OM^2 = AT^2 + AP^2.$$

Similarly $\quad PN^2 = AP^2 + AT'^2.$

Therefore $\quad PM^2 = PN^2.$

37. *Def.*—The straight line, which is the locus of the points from which tangents drawn to two circles are equal, is the **radical axis** of the circles.

38. The three radical axes of three circles meet in a point.

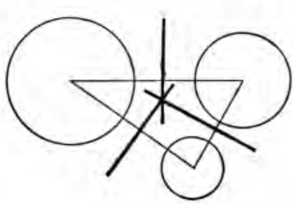


FIG. 38

Hint.—The tangents drawn to the three circles from the point of intersection of two of the radical axes are equal; hence the third radical axis must pass through the point.

39. The difference between the squares of the tangents drawn from any point to two circles is equal to twice the product of the distance of the point from the radical axis by the distance between the centres of the circles.

Hint.—Let C be the centre of OO', AB the radical axis, PR the perpendicular from P on OO'.

$$PT^2 - PT'^2 = (PO^2 - OT^2) - (PO'^2 - O'T'^2).$$
$$PO^2 - PO'^2 = OR^2 - O'R^2 = 2OO'.CR.$$
$$OT^2 - O'T'^2 = OA^2 - O'A^2 = 2OO'.AC.$$

Therefore $$PT^2 - PT'^2 = 2OO'.AR.$$

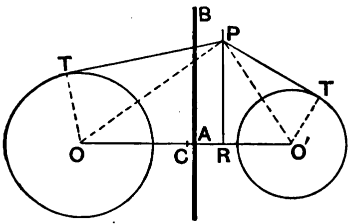


FIG. 39

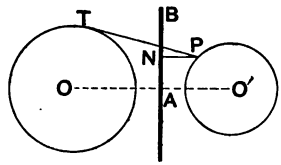


FIG. 40

40. *Cor.*—The square of the tangent drawn from a point on one circle to another circle is equal to twice the product of the distance between the centres of the circles by the distance of the point from their radical axis.

41. *Def.*—A system of circles such that some line is a radical axis common to every pair of circles of the system is a **coaxal system.**

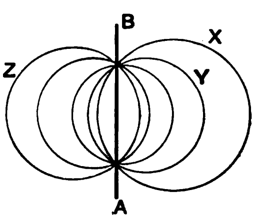


FIG. 41 (1)

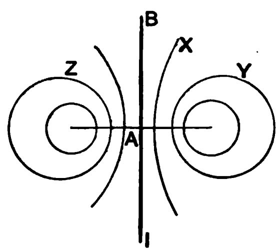


FIG. 41 (2)

Thus if AB is the radical axis of the circles X and Y, X and Z, Y and Z, etc., the system of circles is coaxal.

42. To describe a system of circles coaxal with two given circles.

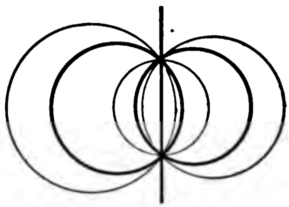


FIG. 42 (1)

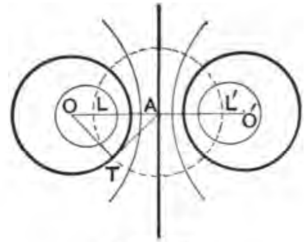


FIG. 42 (2)

(1.) If the circles intersect.

Hint.—The common chord is the radical axis, and any circle through the points of intersection of the two circles is coaxal with them.

(2.) If the circles do not intersect.

Hint.—Let A be the intersection of the radical axis with the line of centres. About A with a radius equal to the tangent AT, describe a circle. This circle will cut the given circles orthogonally. Any circle which has its centre on OO' and is cut orthogonally by this circle is coaxal with the given circles. §§ 37, 41

In a system of coaxal circles which do not intersect, the circles grow smaller as the points in which they are cut by the orthogonal circle approach more nearly the line of centres. The points in which the orthogonal circle cuts the line of centres may be considered as circles of indefinitely small radius, the limiting circles of the system.

Def.—These points are called the **limiting points** of the system.

Thus in Fig. 42 (2) the limiting points are L, L'.

43. From the last article it follows that there are two forms of coaxal circles; in the one the circles intersect and there are no limiting points, in the other the circles do not intersect and there are limiting points.

44. The following special cases of the theorem of § 39 are of importance.

(1.) The square of the distance from any point P of a given circle of a coaxal system to either of the limiting points of the system is proportional to the distance of P from the radical axis.

(2.) If three circles are coaxal, the tangents drawn from any point of the first to the other two are in a given ratio.

(3.) If tangents drawn from a variable point to two given circles are in a given ratio, the locus of the point is a circle coaxal with the given circles.

45. A system of coaxal circles can be described such that each circle will cut orthogonally all the circles of a given coaxal system.

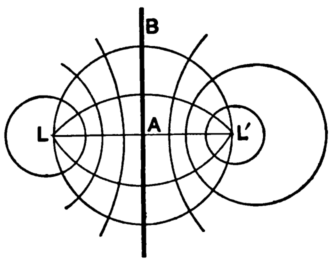


FIG. 45

Hint.—The limiting points of the one system will be the points of intersection of the circles of the other.

Def.—The two systems are called **orthogonal systems** of coaxal circles.

46. *Exercise.*—If a system of circles is cut orthogonally by two circles it is a coaxal system.

47. The inverse of a system of concentric circles is a coaxal system.

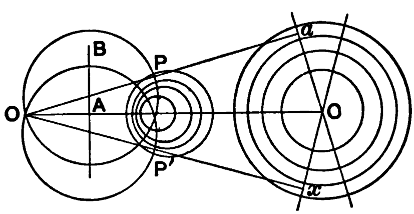


FIG. 47

Hint.—Two straight lines through the centre of the concentric system will invert into two circles cutting the circles of the inverse of the concentric system orthogonally. §§ 34, 46

48. *Remark.*—A system of straight lines passing through a point is a system of intersecting coaxal circles; the other point of intersection is at infinity. A system of concentric circles is a system of non-intersecting coaxal circles; the centre is one of the limiting points, the other limiting point is at infinity.

49. Lines drawn through either of the points of intersection of a system of intersecting coaxal circles are divided proportionally by the circles.

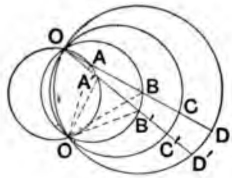


FIG. 48

Hint.—To prove $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \text{etc.}$

Angle $OAO' = OA'O'$; angle $OBO' = OB'O'$. § 201, p. 96

Hence triangle $AO'B$ is similar to triangle $A'O'B'$.

THE STEREOGRAPHIC PROJECTION

50. The stereographic projection furnishes a useful and interesting application of the principles of inversion and coaxal circles.

It has been shown in § 31 that if a sphere be inverted from a point on itself, the inverse is a plane. The result of such an inversion is

FIG. 50 (1)

FIG. 50 (2)

the **stereographic projection** of the sphere. In this projection any figure on the sphere is represented by the figure on the plane into which it inverts; in the inversion, angles of the figure on the sphere and the corresponding angles of its projection on the plane are equal.

The stereographic projection may also be defined as follows: Suppose a transparent sphere have opaque meridians, parallels of latitude, and other lines or figures drawn upon it. The stereographic projection is the picture of these lines and figures obtained if a photographic lens have its optical centre on the surface of the sphere. Or, it is the shadow cast upon a plane without the sphere, if a point of light be at the farther extremity of the diameter perpendicular to this plane. Again, if a line be drawn from the extremity of a diameter of the sphere to any point on the surface of the sphere, its intersection with a plane perpendicular to the diameter is the stereographic projection of this point.

Thus Figures (1), (2), (3), show three forms of the stereographic projection upon a diametral plane. Any plane parallel to this diametral plane would serve as well, and the figures upon the two planes would be similar. The centre of inversion is at E in each case.

Exercise.—Prove by aid of the triangles PRS, QRS of Fig. (4), that the assumption that angles are preserved in this projection is correct. Prove also from Fig. (4) that a circle projects into a circle.

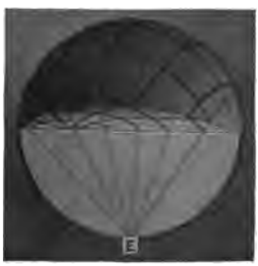


FIG. 50 (3)

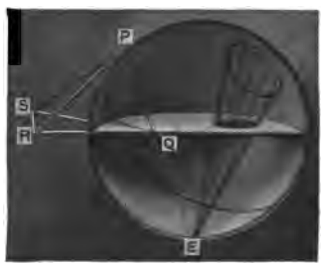


FIG. 50 (4)

The **equatorial** stereographic projection is that obtained if the centre of inversion is at one of the poles of the sphere; it is shown in Fig. (1) and Fig. (5). The parallels of latitude are represented by concentric circles of which the centre is the opposite pole, and the meridians by straight lines through this centre.

The **meridional** stereographic projection is that obtained if the centre of inversion is on the equator; it is shown in Fig. (2) and Fig. (6). The parallels of latitude are represented by a system of coaxal circles of which the poles are the limiting points, and the equator the radical axis. The meridians are represented by a system of intersecting coaxal circles of which the poles are the points of intersection.

The **horizontal** stereographic projection is that obtained if the centre of inversion is on a parallel of latitude other than the equator. It is shown in Fig. (3) and Fig. (7). The parallels of latitude invert into a system of non-intersecting coaxal circles; the poles inverting into the limiting points, and the parallel through the centre of inversion into the radical axis. The meridians invert into a system of intersecting coaxal circles, the poles inverting into their points of intersection.

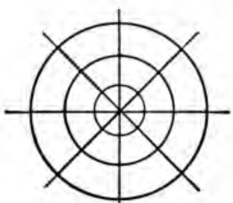


FIG. 50 (5)

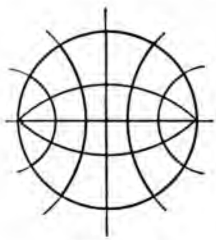


FIG. 50 (6)

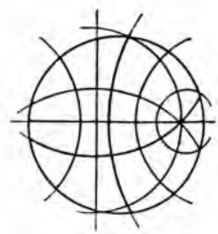


FIG. 50 (7)

It has already been shown that a system of concentric circles and a system of straight lines passing through their centre invert into orthogonal systems of coaxal circles. In accordance with this principle the equatorial projection can be inverted into either the meridional or any desired horizontal projection by properly choosing the centre of inversion; pictures of this inversion, as performed by one of the linkages before described, are shown in Figs. (8) and (9).* Moreover, the meridional projection can be inverted into any desired form of the horizontal in the same manner. It is possible then by a proper choice of the centre of inversion to invert any form of the stereographic projection into any other form desired.

* "Report of the British Association," 1884.

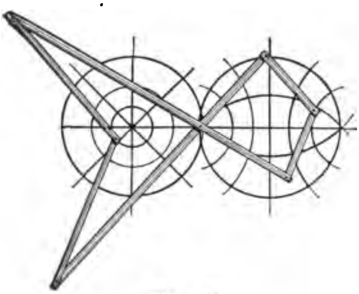


FIG. 50 (8)

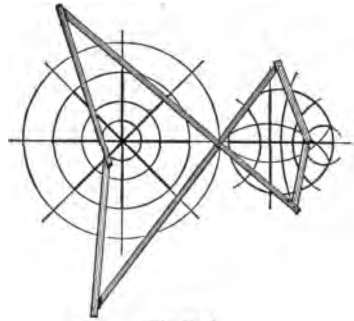


FIG. 50 (9)

POLES AND POLARS

51. *Def.*—If a point is taken on the radius of a circle and another point on the same radius produced, so that the product of their distances from the centre is equal to the square of the radius, each is the **pole** of the line (its **polar**) drawn through the other perpendicular to the radius.

Thus if $OP.OQ=R^2$ the point P is the pole of the line QS, and the line QS is the polar of the point P with respect to the circle X.

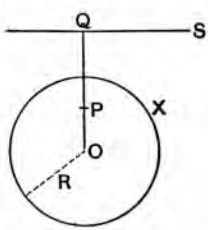


FIG. 51

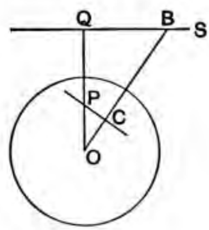


FIG. 52

52. If a line passes through a given point, the pole of the line is on the polar of the point.

Hint.—Let P be the given point, PC the line, QS the polar of P.

Draw OC perpendicular to PC. Since $OC.OB=OP.OQ$, B is the pole of PC.

53. *Cor.*—The line joining two points is the polar of the intersection of their polars; and the point of intersection of two lines is the pole of the line joining their poles.

It follows that the poles of lines which meet in a point are in a straight line, and the polars of points which are in a straight line meet in a point.

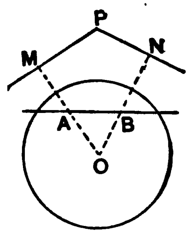


FIG. 53 (1)

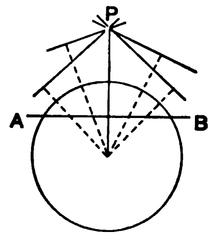


FIG. 53 (2)

Thus if PM and PN [Fig. 53 (1)] are the polars of A and B, AB is the polar of P, and if A and B are the poles of PM and PN, P is the pole of AB.

If several lines meet in a point P [Fig. 53 (2)] their poles are in a straight line AB, and vice versa.

54. The locus of the intersection of tangents to a circle, drawn at the extremities of a chord which passes through a given point, is the polar of the point.

Hint.—Let P be the given point, Q a point on OP such that $OP.OQ=R^2$, and B the intersection of the tangents at the extremities of TT'.

By right triangles $OC.OB=OT^2$.

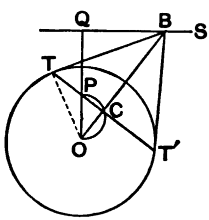


FIG. 54

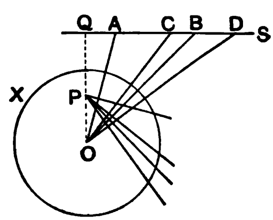


FIG. 55

Hence C inverts into B with respect to the centre of inversion O. But the locus of C is a circle on OP as diameter.

Therefore the locus of B is the straight line QS perpendicular to OQ. § 25

55. If four points on a straight line form a harmonic system, their four polars form a harmonic pencil.

Hint.—Let $ABCD$ be harmonic points on the line QS, and P the pole of QS with respect to the circle X.

OA, OB, OC, OD form a harmonic pencil. Also the polars of the four points A, B, C, D pass through P and are respectively perpendicular to the rays of this harmonic pencil.

Hence the four polars form a pencil which is equiangular with the pencil $(O.ABCD)$ and therefore harmonic.

56. A line cutting a circle and passing through a fixed point is cut harmonically by the circle, the point, and the polar of the point.

Hint.—Let P be the fixed point, LC its polar, and PM the line cutting the circle.

Since $PO.PC=PA.PB=PM.PN$, a circle may be circumscribed about the quadrilateral $OCNM$.

Hence angle $OCM=OMN=PCN$; then CP and CL are the external and internal bisectors of the angle MCN. Therefore P, K, N, M are harmonic points. § 334, p. 151

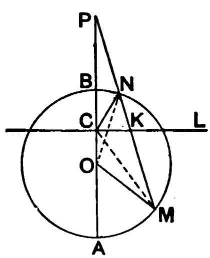


FIG. 56

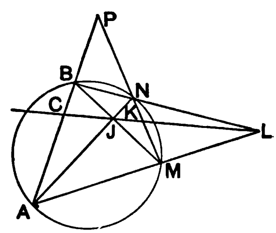


FIG. 57

57. A method of drawing the polar of a given point follows from § 56.

Hint.—Draw the secants PA, PM; also draw AM, AN, BM, BN.

The line through L and J is then the polar of P.

For $LMAJNB$ is a complete quadrilateral; and A, B, C, P, and M, N, K, P, are therefore two systems of harmonic points. § 14

NINE POINTS CIRCLE

58. The circle through the middle points of the sides of a triangle passes through the feet of the perpendiculars from the opposite vertices, and through the middle points of the segments of the perpendiculars included between their point of intersection and the vertices.

Hint.—Let ABC be the triangle, L, M, N the middle points of the sides, O the intersection of perpendiculars, X the middle point of CO.

MX is parallel to AP and consequently perpendicular to ML. Hence a circle on LX as diameter passes through M. For a similar reason it passes through N.

Since LSX is a right angle, the circle passes through S. The circle on MY as diameter must coincide with this circle since it passes through the points L, M, N. Hence the circle also passes through P, etc.

Therefore the circle passes through L, M, N, P, R, S, X, Y, Z.

59. *Def.*—This circle is the **nine points circle** of the triangle.

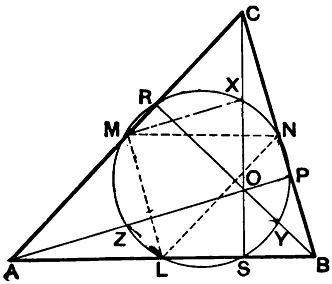


FIG. 58

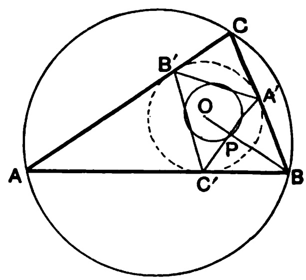


FIG. 60

60. The circumscribing circle of a triangle can be inverted into the nine points circle of the triangle formed by joining the points in which the inscribed circle of the original triangle touches the sides. The centre of inversion is the centre of the inscribed circle; the constant of inversion is equal to the square of its radius.

Hint.—Since $OP.OB = OC'^2$, etc., the vertices A, B, C invert into the middle points of the sides of the triangle.

Hence the circle through A, B, C inverts into a circle through the middle points of the sides of the triangle $A'B'C'$. § 27

PERSPECTIVE

61. *Def.*—Two figures are in **perspective** if the lines joining their corresponding points meet in a common point, the **centre of perspective.**

If the figures are in the same plane they are in **plane perspective.**

Thus if in Fig. (1) lines Aa, Bb, Cc meet in a point O, the triangles ABC, abc are in perspective.

62. If two triangles are in perspective their corresponding sides intersect in points which are in a straight line.

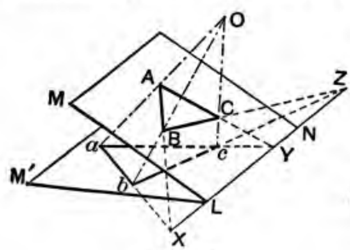


FIG. 62 (1)

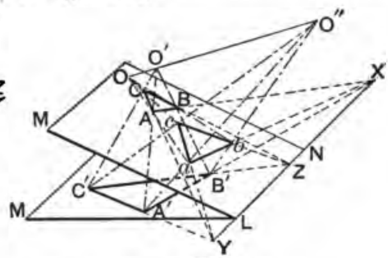


FIG. 62 (2)

(1.) If the triangles are in different planes.

Hint.—Let O be the centre of perspective of ABC, abc.*

Since AB and ab are both in the plane AOB they must meet; since AB is in the plane MN and ab in the plane $M'N$, the point of meeting must be in LN the line of intersection of these planes.

(2.) If the two triangles are in the same plane.

Hint.—Draw any line $OO'O''$ not in the plane of the triangles through the centre of perspective. From any two points O', O'' on this line draw lines through the vertices of the triangles.

$O'A$ and $O''a$ meet in a point A' because both are in the plane $O''OA$;

Thus both the triangles ABC and abc are projected into $A'B'C'$; hence, their corresponding sides meet on the line of intersection of the plane MN with the plane of $A'B'C'$.

63. *Exercise.*—If two polygons are in perspective their corresponding sides meet in points which are in a straight line.

* If MN be a transparent plane and a point of light be at O, the shadow cast upon the plane $M'N$ by the triangle ABC is the triangle abc.

64. *Def.*—The line on which the corresponding lines of two figures in perspective meet is the **axis of perspective** of the figures.

65. Conversely, if the corresponding sides of two plane triangles intersect in points on a straight line, the triangles are in perspective.

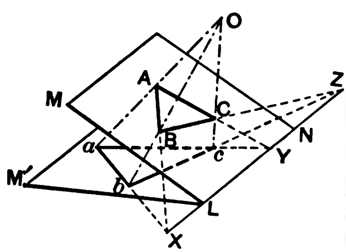


FIG. 65 (1)

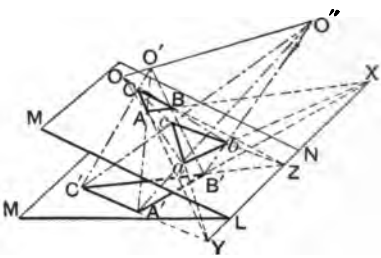


FIG. 65 (2)

(1.) If the triangles are not in the same plane.

Hint.—If AB and ab meet at X, Aa and Bb are both in the plane AXa, and must therefore meet.

Hence Aa, Bb, Cc intersect in pairs, and since they are not all three in the same plane, must therefore meet in a point.

(2.) If the triangles are not in the same plane.

Hint.—Pass any plane through the line in which the corresponding sides meet and construct in it a triangle in perspective with each of the given triangles [§ 62 (2)]. The line through the centres of perspective, O', O'', thus found will meet Aa, Bb, Cc. Therefore Aa, Bb, Cc meet in a point.

66. If three triangles are in perspective two by two, and have the same axis of perspective, their three centres of perspective are in a straight line.

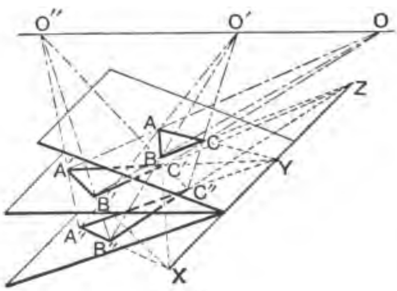


FIG. 66

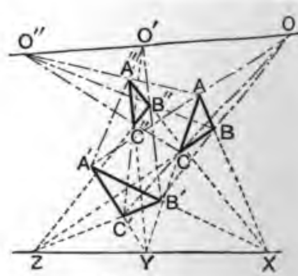


FIG. 66

Hint.—Let ABC, $A'B'C'$, $A''B''C''$ be the triangles, and X, Y, Z the points in which their corresponding sides meet.

The triangles $AA'A''$, $BB'B''$ are in perspective from the centre X.

Hence the intersections of their corresponding sides are in a straight line. But these intersections are the centres of perspective of the original triangles.

67. *Cor.*—If three triangles are in perspective two by two and have the same axis of perspective, the three triangles formed by joining the corresponding vertices of these triangles are also in perspective two by two and have the same axis of perspective; and the axis of perspective of either system of triangles passes through the centres of perspective of the other system.

68. If three triangles are in perspective two by two and have the same centre of perspective, their three axes of perspective meet in a point.

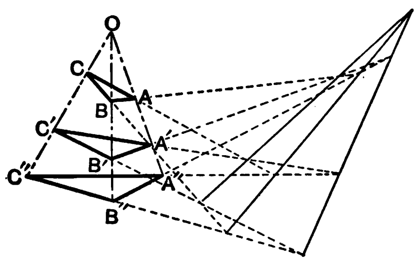


FIG. 68

Hint.—Let ABC, $A'B'C'$, $A''B''C''$ be the triangles and O their centre of perspective.

The triangles formed by the lines AB, $A'B'$, $A''B''$ and by the lines AC, $A'C'$, $A''C''$ are in perspective, since their corresponding sides meet on the line AA'. Therefore the lines joining their corresponding vertices meet in a point.

69. *Cor.*—If three triangles which are in perspective two by two have the same centre of perspective, the three triangles formed by the corresponding sides of these triangles are also in perspective two by two and have the same centre of perspective; and the three axes of perspective of either system meet in the centre of perspective of the other system.

70. *Exercise.*—Extend the theorems of §§ 66 and 68 to figures other than triangles.

DUALITY

71. If the polar of each point and the pole of each line of a figure be taken, a second figure is formed having a peculiar relation to the first and called its **reciprocal.**

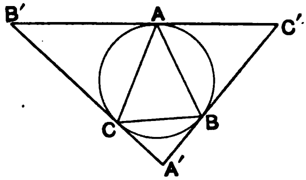


FIG. 71

Thus the triangle $A'B'C'$ is the reciprocal of ABC. The sides of $A'B'C'$ are the polars of the vertices of ABC; the vertices of $A'B'C'$ are the poles of the sides of ABC.

To a point of the first, corresponds a line of the second.

To a line of the first, corresponds a point of the second.

To points in a straight line in the first, correspond lines through a point in the second. § 53

To lines through a point in the first, correspond points in a straight line in the second. § 53

It follows from these relations, that from a theorem concerning the points and lines of a figure, a reciprocal theorem concerning the lines and points of the reciprocal figure can be inferred.

72. *Def.*—The principle upon which these relations between a figure and its reciprocal depend is called the **principle of duality.**

73. The principle of duality in a plane is not necessarily derived from the consideration of poles and polars. A plane figure may be looked upon as composed either of points and the lines joining them, or of lines and their points of intersection, so that the point and line are elements correlative to each other; the relations between reciprocal figures which have already been obtained would follow from this conception.

74. Neither is the principle confined to plane figures; in the same way figures in space may be considered as composed either of points or of planes, so that in the geometry of space the point and plane are elements correlative to each other.

It follows, that for reciprocal figures in space:

To a point in the first, corresponds a plane in the second.

To a plane in the first, corresponds a point in the second.

To points in a plane in the first, correspond planes through a point in the second, and vice versa.

To points in a straight line in the first, correspond planes through a straight line in the second, and vice versa.

Remark.—In the geometry of space the straight line is correlative to itself.

75. Examples of reciprocal theorems of plane geometry.

1. Two points determine a straight line.	1. Two straight lines determine a point, their point of intersection.
2. If the points of intersection of the corresponding sides of two triangles are in a straight line, the lines joining the corresponding vertices of the triangles meet in a point. § 65	2. If the lines joining the corresponding vertices of two triangles meet in a point, the corresponding sides of the triangles intersect in points which are in a straight line. § 62
3. If three triangles are in perspective two by two and have the same centre of perspective, their three axes of perspective meet in a point. § 68	3. If three triangles are in perspective two by two and have the same axis of perspective, their three centres of perspective are in a straight line. § 66

76. Examples of reciprocal theorems of the geometry of space.

1. A straight line and a point determine a plane.	1. A straight line and a plane determine a point, the point in which the line meets the plane.
2. Three points not in the same straight line determine a plane.	2. Three planes which do not pass through the same straight line determine a point.
3. Two straight lines which meet in a point are in the same plane.	3. Two straight lines which are in the same plane meet in a point.

ANHARMONIC SECTION

77. *Def.*—If A, B, C, D are four points taken in order on a straight line, any one of the following six ratios,

$$\frac{AB.CD}{AD.CB}, \quad \frac{AD.BC}{AC.BD}, \quad \frac{AC.DB}{AB.DC},$$
$$\frac{AD.CB}{AB.CD}, \quad \frac{AC.BD}{AD.BC}, \quad \frac{AB.DC}{AC.DB},$$

is an anharmonic ratio of the points A, B, C, D.

FIG. 77

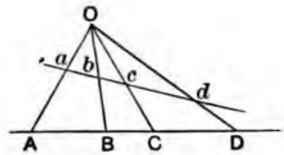


FIG. 78

78. If a pencil of four rays cuts two transversals, each anharmonic ratio of the four points of intersection with one transversal is equal to the corresponding ratio of the four points of intersection with the other transversal.

Hint.—To prove $\dfrac{AB.CD}{AD.CB} = \dfrac{ab.cd}{ad.cb}$, etc.

Compare with § 13.

79. *Cor.* 1.—Anharmonic ratios are preserved in perspective.

80. *Def.*—It follows from § 78 that the anharmonic ratios of a pencil of four rays may be defined as the anharmonic ratios of its four points of intersection with a transversal. § 78

81. *Cor.* 2.—If the corresponding rays of two pencils meet on a common transversal, the pencils are equal, that is, have equal anharmonic ratios.

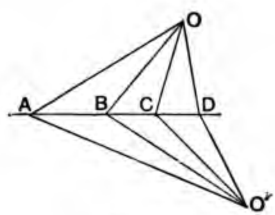


FIG. 81

82. *Cor.* 3.—If two pencils are equal, have a common vertex, and three rays of the first coincide with three rays of the second, the fourth ray of the first coincides with the fourth ray of the second.

83. *Exercise.*—If two pencils have their vertices on a circle and their corresponding rays intersect in points on the circle, the pencils are equal.

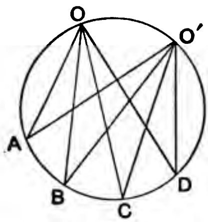


FIG. 83

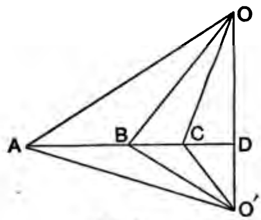


FIG. 84

84. If two equal pencils have a common ray, the intersections of the three remaining pairs of corresponding rays are in a straight line.

Hint.—Employ the method of reductio ad absurdum.

85. *Exercise.*—Prove by means of § 84 that if two triangles are in plane perspective, the intersections of their corresponding sides are in a straight line.

86. (PASCAL'S THEOREM.) If a hexagon is inscribed in a circle, the intersections of the opposite sides are in a straight line.

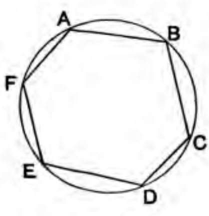


FIG. 86 (1)

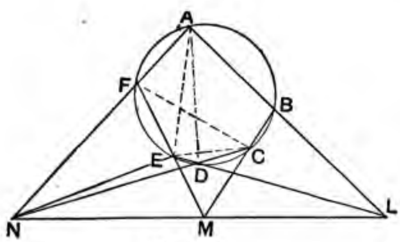


FIG. 86 (2)

Hint.—The opposite sides are the 1st and 4th, 2d and 5th, 3d and 6th. Let L, M, N be the intersections of the opposite sides.

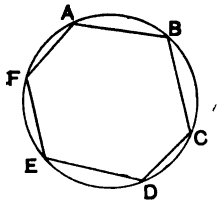


FIG. 86 (1)

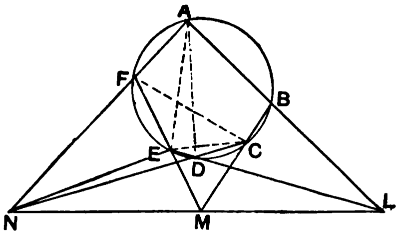


FIG. 86 (2)

Pencil $\left\{N.AEDL\right\} = \left\{A.NEDL\right\}$ by § 81, $= \left\{C.FEDB\right\}$ by § 83,
$= \left\{N.AEDM\right\}$ by § 81.

Therefore L, M, N are in a straight line. § 82

Remark.—This theorem is true of any of the sixty hexagons which can be constructed with six given points as vertices.

87. *Exercise.*—If six points are three by three on two straight lines, the intersections of the opposite sides of a hexagon of which these points are the vertices are in a straight line.

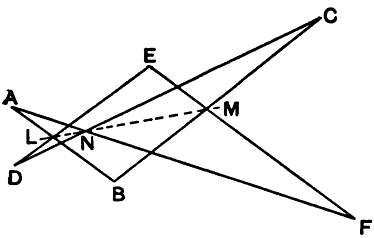


FIG. 87

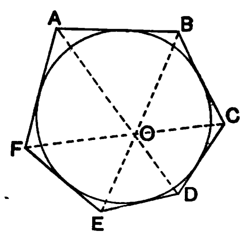


FIG. 88

88. (BRIANCHON'S THEOREM.) If a hexagon is circumscribed about a circle, the three lines joining the opposite vertices meet in a point.

Hint.—The vertices of the circumscribed hexagon are the poles of the sides of an inscribed hexagon. Therefore this theorem may be inferred from § 86 by the principle of duality.

89. *Exercise.*—If four points are in a straight line, their anharmonic ratio is equal to the anharmonic ratio of their four polars.

Hint.—Compare with § 55.

INVOLUTION

90. *Def.*—If the distances of several points, A, A', etc., in a straight line from a point O in that line, are connected by the relation

$$OA.OA'=OB.OB'=OC.OC'=$$

the points form a **range in involution.**

91. If six points form a range in involution, the anharmonic ratios of any four of the points are equal to the anharmonic ratios of their four conjugates.

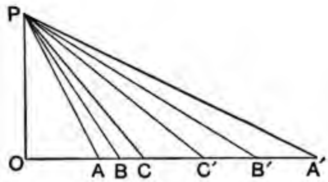


FIG. 91

Hint.—At O erect a perpendicular $OP=\sqrt{OA.OA'}$. Then OP is tangent to the circle described through A, A', P. § 321, p. 145

Hence angle $OPA=OA'P$; likewise angle $OPB=OB'P$, etc.

Therefore angle $APB=A'PB'$, etc.; that is, the angles of the pencil of four rays $\{P.AA'BC\}$ are equal to the angles of the pencil $\{P.A'AB'C'\}$.

The anharmonic ratios of the points A, A', B, C are consequently equal to the anharmonic ratios of the points A', A, B', C'.

92. *Cor.*—The anharmonic ratios of four points in a straight line are equal to the anharmonic ratios of their inverses, if the centre of inversion is on this line.

93. *Def.*—A pencil of which the rays pass through the points of a range in involution is a **pencil in involution.**

ANTIPARALLELS

94. *Def.*—If two lines are such that the inclination of the first to one side of an angle is equal to the inclination of the second to the other side of the angle, the lines are **antiparallel** to each other with respect to the angle.

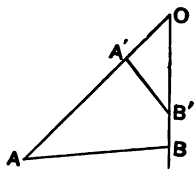


FIG. 94

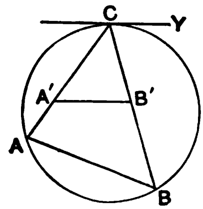


FIG. 95

95. An antiparallel to a side of a triangle with respect to the opposite angle is parallel to the tangent to the circumscribing circle drawn at the vertex of that angle.

Hint.—Angle $YCB = CAB = CB'A'$.

96. *Exercise.*—The lines joining the feet of the perpendiculars of a triangle are antiparallel to the sides with respect to the opposite angles.

THE GEOMETRICAL AXIOMS

Plane, Spherical, and Pseudo-Spherical Geometries

97. The geometrical axioms in the Introduction of this Geometry really define the surface on which the theorems of plane geometry are true. This surface is the plane. The axioms also hold true of any surface into which the plane can be bent without stretching, such as the cylinder or cone, provided the definitions of a straight line and parallel lines be modified to apply to these surfaces.

98. A sheet of paper may be wrapped about a pencil to form a cylindrical surface; every layer of the paper forms a different part of the surface, and two points that lie in different layers one above the other are separated by the distance which must be traversed to get from one to the other without piercing the paper—that is, by the distance they would be separated in the plane if the paper were unrolled.

99. The geometrical axioms are—

(*a*.) **Straight-line axiom.**—Through every two points there is one and only one straight line.

A straight line of any surface may be defined as the shortest line lying

wholly in the surface which can be drawn between two of its points. Thus, arcs of great circles are the straight lines of the *spherical* surface.

(*b*.) **Parallel axiom.**—Through a given point there is one and only one straight line parallel to a given straight line.

Parallel lines of a surface may be defined as straight lines of that surface which meet at infinity.

(*c*.) **Superposition axiom.**—Any figure in a plane may be freely moved about in the plane without change of size or shape.

This axiom as modified would read:

"Any figure of a surface may be freely moved about in that surface without change of size or shape;" that is, would conform to any portion of the surface without stretching.

100. The plane and the surfaces into which it can be bent—the surfaces upon which these axioms hold true—are surfaces of zero curvature.*

101. If the surface or covering of a sphere be detached from the sphere any surface into which it can be bent without stretching is a surface of constant positive curvature. The geometry of such a surface is called **spherical geometry.**

102. The superposition axiom is true for the spherical surface.

103. The straight-line axiom is true for the spherical surface unless the two points are extremities of a diameter of the sphere, in which case an infinite number of straight lines can be drawn between them.

104. There can be no parallel axiom, for on the sphere any two straight lines meet each other at a finite distance.

105. In Book VIII. the spherical geometry is developed, not from the axioms which are true on the covering of a sphere independent of the sphere itself, but by considering this covering as belonging to the body in space. This is entirely unnecessary; the spherical surface may be regarded as an independent surface which has no relation to the plane, the straight line, or space. Its geometry may be developed entirely from the axioms which apply to it, just as the geometry of the plane is developed from its axioms.

* The geometry of such surfaces is called Euclidean Geometry because Euclid first formally stated the axioms as the basis of a geometry.

106. All the theorems of "solid geometry" which relate merely to the *surface* of the sphere would be obtained in this way.

107. Some of the important differences between spherical geometry and plane geometry are that in spherical geometry—

(*a*.) All theorems involving parallel lines are lacking.

(*b*.) The sum of the angles of a triangle is greater than two right angles.

(*c*.) Figures cannot be similar.

(*d*.) The area of a polygon is measured by the sum of its angles—that is, by its spherical excess.

108. If the surface or covering of a pseudo-sphere be detached from the pseudo-sphere any surface into which it can be bent without stretching is a surface of constant negative curvature. The geometry of such a surface is called **pseudo-spherical geometry.**

109. The straight-line axiom and the superposition axiom are true of the pseudo-spherical surface.

110. Through a given point of the pseudo-spherical surface two straight lines can be drawn to meet a given straight line at infinity, one meeting it at infinity in each direction. Consequently on this surface the following must be substituted for the parallel axiom:

Through a given point two straight lines can be drawn parallel to a given straight line.

PLANE SPHERICAL PSEUDO-SPHERICAL

On this surface two lines perpendicular to the same straight line diverge. The appearance of lines perpendicular to the same line on the plane, spherical, and pseudo-spherical surfaces respectively is shown in the above pictures.

111. Pseudo-spherical geometry can be built up from the axioms which are true on the pseudo-spherical surface. Some of the important differences between it and plane geometry are, that in pseudo-spherical geometry—

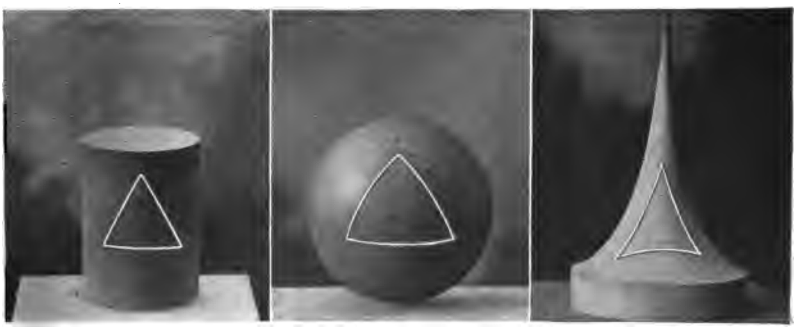
(*a.*) Theorems which assume that non-parallel lines must meet are not true.

(*b.*) Theorems involving parallelism must conform to the parallel axiom for a pseudo-spherical surface.

(*c.*) The sum of the angles of a triangle is less than two right angles.

(*d.*) Figures cannot be similar.

(*e.*) The area of a triangle is measured by two right angles less the sum of its angles—that is, by its pseudo-spherical deficiency.



CYLINDER SPHERE PSEUDO-SPHERE

112. *Remark.*—The circumference of a circle on the plane surface $=2\pi r$; on the spherical surface the circumference is less than $2\pi r$; on the pseudo-spherical surface the circumference is greater than $2\pi r$. The relation of the areas of circles on the three surfaces is the same.

113. There are many theorems which are true in the three kinds of geometry, such as—

The sum of the two adjacent angles which one straight line makes with another straight line is equal to two right angles.

Every point in the perpendicular erected at the middle of a straight line is equally distant from the extremities of that line.

If two angles of a triangle are unequal the sides opposite to them are unequal and the greater side is opposite the greater angle.

The three bisectors of the angles of a triangle meet in a point.

NOTE.—The pseudo-sphere, page 527, is generated by revolving the curve whose equation is

$$y = a \log \frac{a + \sqrt{a^2 - x^2}}{x} - \sqrt{a^2 - x^2}$$

about its y-axis. The radius of the base of the pseudo-sphere is a.

TABLE OF MEASURES AND WEIGHTS

English Measures

LENGTH

12 inches (in.)	= 1 foot (ft.).
3 feet	= 1 yard (yd.).
$5\frac{1}{2}$ yards	= 1 rod (rd.).
4 rods	= 1 chain (ch.).
80 chains	= 1 mile (m.).
1 yard	= .9144 meter.
1 mile	= 1.6093 kilometers.

SURFACE

144 sq. inches	= 1 sq. foot.
9 sq. feet	= 1 sq. yard.
$30\frac{1}{4}$ sq. yards	= 1 sq. rod.
160 sq. rods	= 1 acre.
640 acres	= 1 sq. mile.
1 sq. yard	= 0.8361 sq. meter.
1 acre	= 0.4047 hectare.

VOLUME

1728 cu. inches	= 1 cu. foot.
27 cu. feet	= 1 cu. yard.
128 cu. feet	= 1 cord (cd.).
1 cu. yard	= 0.7646 cu. meter.
1 cord	= 3.625 steres.

ANGLES

60 seconds (″)	= 1 minute (′).
60 minutes	= 1 degree (°).
90 degrees	= 1 right angle.

CIRCLES

360 degrees = 1 circumference.

$\pi = 3.1416 = \text{nearly } 3\frac{1}{7}$.

CAPACITY

1 liq. gal. = 3.785 liters = 231 cu. in.
1 dry gal. = 4.404 liters = 268.8 cu. in.
1 bushel = 0.3524 hkl. = 2150.42 cu. in.

AVOIRDUPOIS WEIGHT

16 ounces (oz.)	= 1 pound (lb.).
100 lbs.	= 1 hundredweight (cwt.).
20 hundredweight	= 1 ton (T.).

1 pound = .4536 kilo. = 7000 grains.
1 ton = .9071 tonneau.

Metric Measures

LENGTH

10 millimeters (mm.)	= 1 centimeter (cm.).
10 centimeters	= 1 decimeter (dcm.).
10 decimeters	= 1 meter (m.).
10 meters	= 1 dekameter (dkm.).
10 dekameters	= 1 hektometer (hkm.).
10 hektometers	= 1 kilometer (km.).
1 meter	= 39.37 inches.
1 kilometer	= 0.6214 mile.

SURFACE

100 sq. millimeters	= 1 sq. centimeter.
100 sq. centimeters	= 1 sq. decimeter.
100 sq. decimeters	= { 1 sq. meter. 1 centare (ca.).
100 centares	= 1 are (a.).
100 ares	= 1 hektare (hka.).
1 sq. centimeter	= 0.1550 sq. inch.
1 sq. meter	= 1.196 sq. yards.
1 are	= 3.954 sq. rods.
1 hektare	= 2.471 acres.

VOLUME

1000 cu. millimeters	= 1 cu. centimeter.
1000 cu. centimeters	= 1 cu. decimeter.
1000 cu. decimeters	= 1 cu. meter.
	= 1 stere (st.).
1 cu. centimeter	= 0.061 cu. inch.
1 cu. meter	= 1.308 cu. yards.
1 stere	= 0.2759 cord.

CAPACITY

100 centiliters (cl.)	= 1 liter (l.).
100 liters	= 1 hektoliter (hkl.)

1 liter = 1.0567 liq. qts. = 1 cu. dcm.

METRIC WEIGHT

1000 grams (gm.)	= 1 kilogram (kilo.).
1000 kilograms	= 1 tonneau (t.).
1 gram	= 15.432 grains.
1 kilogram	= 2.2046 pounds.
1 tonneau	= 1.1023 tons.

INDEX OF DEFINITIONS

[The references are to sections. Ex. refers to Exercises, pp. 463–492. M. G. refers to Modern Geometry, pp. 493–528.]

INDEX OF CONSTRUCTIONS IN PLANE GEOMETRY

INDEX OF RULES OF MENSURATION